

Exact Zero Temperature Correlation Functions for Two-Leg Hubbard Ladders and Carbon Nanotubes

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(February 1, 2008)

Motivated by recent work of Lin, Balents, and Fisher [1], we compute correlation functions at zero temperature for weakly coupled two-leg Hubbard ladders and (N, N) armchair carbon nanotubes. In [1] it was argued that such systems renormalize towards the $SO(8)$ Gross-Neveu model, an integrable theory. We exploit this integrability to perform the computation at the $SO(8)$ invariant point. Any terms breaking the $SO(8)$ symmetry can be treated systematically in perturbation theory, leading to a model with same qualitative features as the integrable theory. Using said correlators, we determine the optical conductivity, the single-particle spectral function, and the I-V curve for tunneling into the system from an external metallic lead. The frequency, ω , dependent optical conductivity is determined exactly for $\omega < 3m$ (m being the fermion particle mass in the $SO(8)$ Gross-Neveu model). It is characterized by a sharp “exciton” peak at $\omega = \sqrt{3}m$, followed by the onset of the particle-hole continuum beginning at $\omega = 2m$. Interactions modify this onset to $\sigma(\omega + 2m) \sim \omega^{1/2}$ and not the $\omega^{-1/2}$ one would expect from the van-Hove singularity in the density of states. Similarly, we obtain the exact single particle spectral function for energies less than $3m$. The latter possesses a delta function peak arising from single particle excitations, together with a two-particle continuum for $\omega \geq 2m$. The final quantity we compute is the tunneling I-V curve to lowest non-vanishing order in the tunneling matrix elements. For this quantity, we present exact results for voltages, $V < (1 + \sqrt{3})m$. The resulting differential conductance is marked by a finite jump at $\omega = 2m$, the energy of the onset of tunneling into the continuum of two particle states. Through integrability, we are able to characterize this jump exactly.

All calculations are done through form-factor expansions of correlation functions. These give exact closed form expressions for spectral functions because the $SO(8)$ Gross-Neveu model is massive: each term in the expansion has an energy threshold below which it does not contribute. Thus, we obtain exact results below certain thresholds by computing a finite number of terms in this series. Previous to this paper, the only computed form-factor of $SO(8)$ Gross-Neveu was the two particle form factor of an $SO(8)$ current with two fundamental fermions. In this paper we compute the set of all one and two particle form factors for all relevant fields, the currents as well as the kinks and fermions.

I. INTRODUCTION

Strongly interacting systems are a focal point of modern solid state physics. The behaviour of these systems is often not captured by approximations, based on various ‘free particle’ non-interacting models, since in the presence of generically strong interactions the physics is typically much richer, exhibiting qualitatively new features. Computational tools to analyze the behaviour of strongly interacting theories are few in number. Amongst these are the tools of integrability, which are available for a number of special $1+1$ -dimensional models. These methods allow for exact and detailed computations of many physically relevant properties of these theories. But as integrable models have to be of a special form, i.e. need to be ‘fine-tuned’, it is often believed that these tools are of little relevance for systems observed in the laboratory, since the latter would never generically be of the particular form necessary for integrability.

However this need not be the case. Recently, in the context of transport experiments on tunneling point contacts in fractional quantum Hall effect devices [2], it has

been pointed out by [3] that when renormalization group (RG) thinking is combined with integrability, a special and ‘fine-tuned’ integrable Hamiltonian may actually describe the behaviour of a realistic system. The Hamiltonian of a generic system may be attracted under RG transformations to an integrable one, which then can be analyzed using the methods of integrability. In this way these powerful methods can be brought to bear on experimentally realizable systems and phenomena, and their predictions be directly compared with experimental data.

In this paper we address another set of examples of a similar spirit. These are quasi one-dimensional interacting electronic systems, of which two prominent experimental realizations are two-leg Hubbard ladders (see for example [4]) and single-walled carbon nanotubes [5].

Two-leg ladders have recently been the focus of much theoretical and experimental activity. At half-filling they are Mott insulators, exhibiting gaps to all excitations, and in particular a spin gap. These are typical examples of ‘spin-liquids’. Upon doping, the gaps to all excitations except for those with charge-two survive [6]. The gapless charge modes induce quasi long-range superconducting

pairing correlations, with approximate d -wave symmetry, reminiscent of underdoped cuprate superconductors.

Carbon nanotubes are novel materials whose mechanical and electronic properties promise potential for new technological applications [5]. They are formed by wrapping graphite sheets into cylinders of nanoscale dimensions. They support electronic excitations, which, for a prominent member of the nanotube family, the armchair (n,n) type, can be described by the same theoretical model as that used for the two-leg Hubbard ladders [7,8]. Even though these systems would be one-dimensional band metals in the absence of interactions, they become Mott insulators at half-filling due to the presence of short-ranged electronic interactions, which play an important role due to their one-dimensional nature. It is these interaction effects that we analyze exactly in this paper using the powerful methods of integrability. After experimental techniques had been developed to fabricate long single-walled nanotubes with high yields in the laboratory, this field of material science has seen an explosive development [9]. Electronic properties can be measured relatively easily by attaching metallic leads [10] or by tunneling into these materials with scanning tunneling microscope (STM) tips. The practical feasibility of such tunneling experiments from an STM tip into an individual single-walled nanotube placed upon a gold substrate (screening long-range Coulomb forces) has recently been demonstrated by J.W.G. Wildöer et al. [9]. Thus, the predictions that we make in this paper, especially those for the tunneling $I(V)$ -curve, have direct bearing on possible future experiments on these materials.

As said, both systems, two-leg ladders and single-walled armchair nanotubes, would be one-dimensional metals in the absence of electron interactions. These are described theoretically (on scales much smaller than the non-interacting band width) by two species of spinful massless Dirac fermions in $(1+1)$ dimensions¹. For the ladder compounds, the two species arise in an obvious way from the two rungs of the ladder, whereas for the nanotubes they arise from the particular band structure of the underlying hexagonal graphite lattice, characterized by two Fermi points in the Brillouin zone [7,8]. These massless Fermi surface excitations interact with short-range interactions whose detailed nature is determined by non-universal microscopic considerations.

A notable observation was made recently by Lin, Balents, and Fisher [1]. These authors argued that within an 1-loop RG any such model with generic, non-chiral, short range interactions flows at half-filling into a theory with an immense symmetry, namely the $SO(8)$ sym-

metric Gross-Neveu model. This model not only has a large $SO(8)$ global symmetry, which encompasses an one-dimensional version of $SO(5)$ recently advocated by S. C. Zhang [11], but in addition has an infinite number of hidden conservation laws, which are a consequence of the integrability of this model. The $SO(5)$ subgroup symmetry is the same as that studied by [12] in the context of lattice models of interacting two-leg ladders. We point out that it would be straightforward to construct a corresponding $SO(8)$ invariant lattice model of a two-leg ladder.

The use of the RG in [1] can be understood in the following sense. Since the analysis in [1] is based on an 1-loop RG, the initial microscopic (bare) interactions must be small enough so that the integrable $SO(8)$ invariant RG trajectory is approached sufficiently closely after a number of RG steps, before leaving the range of validity of the 1-loop RG equations. Whenever this is the case, it is argued that the integrable model is approached independently of the (sufficiently weak) values of the bare interactions. The situation for the 2-leg ladder is thus similar in spirit to that of the point contact device encountered in [3], where only a single operator was relevant, and this relevant operator was integrable. In the latter case all other interactions were irrelevant in the RG sense, and could in principle be treated perturbatively.

The requirement of the RG that the interactions be short-ranged is natural in the case of the Hubbard ladders. However it may not seem so in the case of the carbon nanotubes. Recent theory [13] and experiment [14] have discussed the case where long-ranged Coulomb forces drive Luttinger liquid behaviour in single-walled carbon nanotubes. However we do not have such situations in mind for the paper at hand. Rather we want to consider situations such as those found in the experiments of J.W.G. Wildöer et al. [9] where the long range forces are screened.

Although the restriction to such experiments in the case of the carbon nanotubes places us upon safe ground, it is not inconceivable that experiments where the long-ranged forces are present would nevertheless see behaviour indicative of the $SO(8)$ symmetry. An unscreened force translates into an unusually large bare coupling (in comparison with other bare couplings) in the forward scattering direction. However this does not mean the RG is inapplicable. The RG still indicates a potential enhancement in the symmetry. Because of the large bare coupling, the RG must be run a longer time before any enhancement would be seen but nevertheless an enhancement may well occur at some low energy scale. In terms of the experiments in [14], this would mean that at medium energy scales, Luttinger liquid behaviour would

¹These two species have equal Fermi velocities due to particle-hole symmetry present at half-filling.

predominate, while at much lower energy scales, $SO(8)$ behaviour would be expected. However at current standing, the material science is not advanced to the point where it is possible to accurately probe the very low energy behaviour. But the potential for advancement in this area is ever present.

The RG analysis further requires the bare couplings to be weak. With Hubbard ladder compounds, this condition will not be generically met, although it certainly will not be universally violated. However with (N,N)-armchair carbon nanotubes, the bare couplings are naturally weak. It is one of the hallmarks of the physics of the (N,N) armchair carbon nanotubes that the electrons are delocalized around the circumference of the tube. This in turn leads to a scaling of the effective short-ranged interaction by $1/N$, making it naturally small [7].

It can, however, be questioned on a more fundamental level whether a 1-loop RG adequately describes the system's behaviour. Difficulties with the analysis in [1] take two forms. As a first objection, the authors of [15] point out that an RG flow can imply a symmetry restoration which in fact does not occur. As an example they consider a $U(1)$ symmetric Thirring model,

$$\mathcal{L} = \bar{\Psi}_\alpha \gamma^\mu \partial_\mu \Psi_\alpha + \frac{1}{4} g_\parallel (j_z)^2 + \frac{1}{4} g_\perp [(j_x)^2 + (j_y)^2], \quad (1.1)$$

where $j_a^\mu = \bar{\Psi} \gamma^\mu \sigma_a \Psi$. Although the 1-loop RG equations for this model seems to indicate a generic symmetry restoration to a more symmetric $SU(2)$ case (i.e. $g_\perp = g_\parallel$), this in fact only occurs in a certain region of coupling space. For $\pi - |g_\perp| > -g_\parallel > |g_\perp| > 0$, the $U(1)$ model maps onto the sine-Gordon model with interaction $\cos(\beta\phi)$ [16], where β is given by

$$\beta^2 = 8\pi - 8\mu; \quad \mu = \cos^{-1}[\cos(g_\parallel)/\cos(g_\perp)]. \quad (1.2)$$

The value of β completely characterizes the model. While g_\perp and g_\parallel flow under the RG, the particular combination of these parameters forming β does not. Thus for this particular region of parameter space the model moves no closer to the $SU(2)$ symmetric point under an RG flow. In other regions however (for example $|g_\perp| > |g_\parallel|$), the situation is better; the effect of the anisotropy in the couplings is exponentially suppressed.

However it is reasonably clear that such pessimism is not warranted in the analysis of the RG of [1]. A salient criticism of [15] is that in considering the action of the renormalization group, they fail to consider the consequences of working in the scaling limit. The scaling limit is exactly the limit in which a field theory becomes available. In turn, the scaling limit places constraints upon the possible range of bare couplings consistent with a field theory. In the case of sine-Gordon, the underlying integrability/solvability of the theory allows explicit investigation of this question. It is found that the allowed range is such that even moderate anisotropic deviations are forbidden [17]. The scaling limit, in other words, enforces isotropy. In cases where there are RG flows indicating an enhancement in symmetry, this turns out to be a general phenomena and it leads to an expanded notion of symmetry restoration [17].

On a more concrete level, the breaking of the $SU(2)$ symmetry considered in [15] is a rather special case. Ultimately, the parameter β in the sine-Gordon model is protected under an RG flow by the presence of a quantum group symmetry arrived at by deforming a Yangian symmetry present at the $SU(2)$ point. There is, however, no such known way to deform the Yangian in $SO(8)$ Gross-Neveu. Indeed the natural generalization of the sine-Gordon model to $SO(8)$ is not to $SO(8)$ Gross-Neveu but to an affine Toda $SO(8)$ theory where such a deformation of the Yangian symmetry is possible [18].

Another question that one must ask in looking at the analysis in [15] is how the choice of the symmetry breaking terms affects the symmetry restoration. The sine-Gordon model still possesses a $U(1)$ symmetry. However it is certainly possible to consider perturbations that break this $U(1)$. Such perturbations would destroy the quantum group symmetry of sine-Gordon and thus might lead to symmetry restoration. This would be perhaps closer to the RG analysis of [1] where a large number (nine) of marginal perturbations were included. We in fact consider exactly such a situation in [17] and find that indeed there is symmetry restoration.²

The second objection to the analysis of [1] is its omission of chiral interactions that alter the Fermi velocities

²We note in passing that the authors of [15] consider an anisotropic Gross-Neveu model, a model of direct relevance to the situation at hand. They conclude through a mean-field/large N limit computation that the model is intrinsically anisotropic thus throwing doubts upon the analysis in Lin et al. [1]. There appear to be several severe problems, however, with their analysis. The most telling is the manner in which the bare couplings are scaled in the large N limit. The anisotropic model they consider has three bare couplings. One is chosen to not scale at all, one scales as $1/N$, and the last scales as $1/N^{d/2}$, $d < 1$. With this scaling [15] the model

[19]. Such interactions, although they are absent from the 1-loop RG, likely play a role at higher order. However their effect is less drastic than envisioned in [19]. There a scenario was considered where the invariant RG trajectory of higher symmetry was inherently unstable to perturbations. However the $SO(8)$ RG ray in [1] has a basin of attraction of finite measure. The effect of chiral interactions is to then slightly alter the direction of the ray [20]. In turn the ratio of masses of the various excitations are slightly perturbed away from one.

In taking account of these objections, prudence suggests a modification in the understanding of the RG analysis of [1]. This analysis in [1] and [20] tells us that while the RG flow does not restore an exact symmetry, it leaves us close to the symmetrical situation. In particular, it indicates that while the masses in the actual system may differ from their $SO(8)$ values, they do not wildly diverge. One then understands the $SO(8)$ Gross-Neveu theory, not as precisely representative of the actual system, but in near perturbative vicinity of it, that is, as an excellent starting point about which to perform perturbation theory in the non-integrable interactions breaking $SO(8)$ in much the same spirit as done for a non-critical Ising model in the presence of a magnetic field [21].

In doing perturbation theory about $SO(8)$ Gross-Neveu it is important to emphasize two salient points. Firstly $SO(8)$ Gross-Neveu already contains all the basic features of the strongly interacting electron system. It thus provides a much better starting point for perturbation theory than the two band metal from which it arose. In contrast, perturbation theory about the two band metal could not hope to capture, even qualitatively, characteristics of the interacting system. Secondly, $SO(8)$ Gross-Neveu is a massive theory and so poses none of the attendant problems of perturbation theory with a massless model (as a two band metal is). The perturbative series is well controlled; the perturbed theory is connected in a continuous fashion to $SO(8)$ Gross-Neveu.

In this paper we focus on the integrable model at half-filling and at zero temperature³ and as such this work should be considered as the starting point for the analysis of a perturbed $SO(8)$ Gross-Neveu ladder/nanotube. (We do, however, in Section 3 make a rough sketch of the effect of perturbations on our results. A more detailed analysis awaits further work.) Some consequences of the integrability of the theory have been exploited in [1], notably

those following directly from the structure of the elementary excitations (‘particles’) of the theory. If U is the typical strength of the interactions and t a measure of the bandwidth of the non-interacting model, the Gross-Neveu interactions generate dynamically a single mass scale $m \propto te^{-t/U}$. Due to the integrability, the masses of all excitations are universally related to m , universal mass ratios being known exactly. This, together with the knowledge of the quantum number assignment to these excitations, has been exploited in [1] to make, *inter alia*, a number of predictions for correlations functions, largely of a qualitative nature.

Due to its integrability, an immense amount of information is known about this strongly interacting system. In this paper, we put this information to use, to compute the exact functional form of spectral functions, at zero temperature and on scales below certain thresholds (to be discussed below). To this end we use the form factor approach described in [22]. In particular, we have obtained exact expressions for the optical conductivity, the single-electron spectral function (experimentally accessible via photoemission experiments), as well as the tunneling density of states (experimentally observable for example by measurement of the differential conductance for tunneling from a metallic lead into a nanotube [9]).

We emphasize that our results are exact on energy scales below certain thresholds. In order to appreciate the significance of this, it is important to recall that even in an integrable system, correlation functions cannot in general be computed exactly. However if a mass gap is present for all excitations (such as for the Gross-Neveu model) exact results for spectral functions can be obtained below certain thresholds. This can be easily understood from the basic ideas underlying the form factor approach to correlation functions in integrable systems, which we now briefly review. The key feature of an integrable system is the exact knowledge of a basis of eigenstates of the fully interacting Hamiltonian. At the root of integrability is a well defined notion of “particles”, or “elementary excitations” in the fully interacting system. These particles scatter off each other only with two-body S -matrices, that is, all particle production processes are absent and particle number is conserved. This is due to special conservation laws which exist in an integrable model, preventing the decay of these particles. In this sense, an integrable system is similar to a Fermi liquid.

is anisotropic. However all that was done is to examine a system with a diverging bare anisotropy and conclude that the effective theory is similarly anisotropic. And nowhere is the RG, which in this model indicates symmetry restoration, allowed to act. As such, we believe this example has little bearing on the situation at hand.

³The doped system is examined in [6].

An additional feature is that new particles can arise as bound states of already existing ones. However the total number of different types of particles is finite which makes the system analytically tractable. For the $SO(8)$ Gross-Neveu model these particles consist of fermionic particle states of mass m , namely one octet of fundamental fermions and two octets of kinks. The excitations of the ladder or nanotube, possessing the quantum numbers of the electron, are represented by eight of these kinks. In addition there are 29 bosonic particle states organized as a rank-2 $SO(8)$ anti-symmetric tensor and a singlet, all of mass $\sqrt{3}m$. These are bound states of the fermionic mass- m particles. The exact eigenstates of the interacting Hamiltonian are simply n -particle states, $|n; s_n\rangle$, where s_n collectively denotes the momenta and species labels of the n particles⁴.

The form-factor representation of any correlation function is obtained by inserting a resolution of the identity corresponding to the basis of particle eigenstates. Thus, for an operator, $\mathcal{O}(x, \tau)$, we write the spectral decomposition schematically (τ denotes imaginary time, and T time ordering)⁵

$$G_T^{\mathcal{O}}(x, \tau) = \langle 0 | T(\mathcal{O}(x, \tau) \mathcal{O}^\dagger(0, 0)) | 0 \rangle \\ = \sum_{n=0}^{\infty} \sum_{s_n} \langle 0 | \mathcal{O}(x, 0) | n; s_n \rangle e^{-\tau E_{s_n}} \times \\ \langle n; s_n | \mathcal{O}^\dagger(0, 0) | 0 \rangle, \quad (\tau > 0), \quad (1.3)$$

where E_{s_n} is the energy of the eigenstate, $|n; s_n\rangle$. In an integrable model the matrix elements of the physical operator between the vacuum and the exact eigenstates can in principle be computed exactly from the 2-body S-matrix. However the calculation of these matrix elements, as well as the evaluation of the sums/integrals \sum_{s_n} , becomes increasingly cumbersome as the particle number n becomes large, so that the full expression for the correlation function cannot be evaluated in closed form. Often, however, a truncation of the sum at the level of two or three particle states already provides a good approximation to the full correlation function [23–28]. On the other hand, this truncation is no longer necessary in a massive theory, if one considers the corresponding spectral function. Only eigenstates with a fixed energy, ω , contribute to the spectral function:

$$-\frac{1}{\pi} \text{Im} G_T^{\mathcal{O}}(x, -i\omega + \delta) = \sum_{n=0}^{\infty} \sum_{s_n} \left\{ \langle 0 | \mathcal{O}(x, 0) | n; s_n \rangle \langle n; s_n | \mathcal{O}^\dagger(0, 0) | 0 \rangle \delta(\omega - E_{s_n}) \right. \\ \left. - \epsilon \langle 0 | \mathcal{O}^\dagger(0, 0) | n; s_n \rangle \langle n; s_n | \mathcal{O}(x, 0) | 0 \rangle \delta(\omega + E_{s_n}) \right\}, \quad (1.4)$$

where $\epsilon = \pm$ for fields, \mathcal{O} , that are bosonic/fermionic.

Since in a massive theory the creation of an extra particle in the intermediate exact eigenstate costs a finite amount of energy, the sum in 1.4 is finite. For example, when ω is smaller than the energy of all three-particle states (i.e. when ω is below the three-particle threshold), then only the form factors with one and two particles ($n = 1, 2$) have to be determined. This is what we have done in this paper for all the spectral functions we computed. Thus, the results obtained from the form factor method for spectral functions are exact in massive theories.

The results for the physical quantities we compute, namely the optical conductivity, the single particle spec-

tral function, and the tunneling $I - V$ curve, are summarized in Section 3. The basic features of these results are as follows. The optical conductivity, $\text{Re}[\sigma(\omega)]$ (so called because it would be measured in a reflectivity experiment) has two notable features: a delta function peak at $\omega = \sqrt{3}m$ corresponding to an excitation of one of the 28 rank two tensorial bosonic particles, and a continuum of two particle states beyond $\omega = 2m$ (see Figures 1 and 2). In a free theory, the van-Hove square-root singularity in the density of states at the two particle threshold would lead $\text{Re}[\sigma(\omega)]$ to behave as $1/\sqrt{\omega - 2m}$ ⁶. However the current matrix elements vanish at threshold changing the behaviour to $\sqrt{\omega - 2m}$. This vanishing is ultimately the result of generic interactions becoming strongly renor-

⁴This can be made more explicit by writing, $|n; s_n\rangle = |(p_1, a_1); (p_2, a_2); \dots; (p_n, a_n)\rangle$.

⁵The sum over s_n is meant to include integrals over the momenta of all particles.

⁶This is the result found in [1]. In this work the optical conductivity is computed in the large N limit of $SO(2N)$, in effect an RPA approximation. In this limit the theory consists of four massive but non-interacting Dirac fermions.

malized at low energies [29]. However its exact behaviour as one moves away from threshold can only be extracted through integrability.

As the $SO(8)$ Gross-Neveu model characterizes the low energy behaviour of the Hubbard ladders/carbon nanotubes, we are able to describe the single particle spectral function only near the Fermi points. Here we find two principal features. There is a sharp peak (infinitely so at $T = 0$) describing the single electron contribution to the spectral function. As the system is interacting we expect to find a continuum of higher particle contributions to the spectral function. In physical terms these higher particle contributions would take the form of neutral particle-hole excitations together with an electron. In the Gross-Neveu language said excitations take the form of a kink (akin to the electron) and a fundamental Gross-Neveu fermion (akin to the particle hole excitation). The threshold of this two-particle contribution occurs at an energy, $\omega = 2m$.

One of the quantities most easily measured in experiments on nanotubes is the tunneling density of states. This quantity can be measured by tunneling from a metallic lead, such as an STM tip, into the nanotube. When applying a voltage bias, a current will flow. This current is a non-linear function of the applied bias. To lowest non-vanishing order in the tunneling matrix element, this current can be related to the single-particle spectral function at energy $\omega = V$ (using standard reasoning). This relationship is non-perturbative in the voltage V . Differentiating the so-obtained non-linear current-voltage characteristic, the differential conductance for tunneling into the nanotube is easily seen to equal the spectral function of the single-particle Greens functions. Physically, it is a ‘spectroscopic probe’, or ‘measure’ of the density of states in the fully interacting nanotube at energy V , even though it is not equal to it as the spectral function is equal to the density of states multiplied by the form-factor matrix elements. A look at Figure 5 reveals an interesting novel feature, entirely due to the interactions, and not visible in a band-semiconductor: the differential tunneling conductance develops a jump of a finite magnitude at the two-particle threshold. This might have been expected on physical grounds, since at this energy extra states suddenly become available for current transport. Nevertheless, is it by no means obvious that this would correspond to a discontinuity in the tunneling density of states; the two-particle channel could also open up gradually. This depends on the detailed behaviour of the matrix elements near the threshold, which we have computed exactly from integrability. Our analysis shows that these matrix elements behave in

such a way that there is a finite jump.

The one and two particle form factors have not in general been computed before for the $SO(8)$ Gross-Neveu model. The sole exception is the matrix elements of the current operators with the fundamental fermions [30]. The calculation of the remaining form factors, and in particular those involving the kinks, makes up the technical part of this paper. To compute the form factors, we employ a series of algebraic constraints that arise from consistency with known two body scattering matrices, unitarity, Lorentz invariance, and braiding relations. The first three are common in form factor calculations (see for example [26–28,23–25,30]). The latter is perhaps more unusual [27,22,31]. Braiding relations arise as generalized commutators of fields, ψ^a :

$$\psi^a(x,t)\psi^b(y,t) = R_{cd}^{ab}\psi^d(y,t)\psi^c(x,t), \quad x > y, \quad (1.5)$$

where R_{cd}^{ab} is termed the braiding matrix. They are both indicative of the non-locality of the fields for which we compute form factors and of exotic symmetries. Non-trivial braiding relations are tied to the existence of quantum group symmetries [32]. In this case, the relevant quantum group symmetry is an $SO(8)$ Yangian. This is similar to the situation found in the well-studied sine-Gordon model at its $SU(2)$ point [33], where an $SL(2)$ Yangian symmetry is present.

The $SO(8)$ Yangian is the operative symmetry of the model. The particles in the theory are thus organized in terms of its finite dimensional representations and not $SO(8)$ ’s. There is however a large degree of correspondence between the two sets of representations. The three fundamental eight dimensional representations of $SO(8)$ are also irreducible representations of the Yangian. However the 28 dimensional second rank anti-symmetric tensor representation does not appear as a representation of the Yangian. Rather under the Yangian it is combined with the one-dimensional scalar representation into a single 29 dimensional representation⁷. Thus, unsurprisingly the scalar particle together with the 28 particles of the anti-symmetric tensor share the same mass, $\sqrt{3}m$.

The paper is organized as follows. In Section 2, we review the arguments given by Lin, Balents and Fisher [1] showing how the various massive phases of the Hubbard ladders/nanotubes are related to an $SO(8)$ Gross-Neveu model. In doing so we establish notation. In Section 3, as stated previously, we summarize our results for the optical conductivity, the single-particle spectral function, and the tunneling I-V curve. Sections 4 and 5 are devoted to computing the form factors. For readers uninterested in the details of this analysis, the last part of Section 5

⁷We would like to thank N. MacKay for stressing this fact to us.

provides a summary of results. Specifically, Section 4 reviews the S-matrices and attendant group theory for the $SO(8)$ Gross Neveu model, while Section 5 presents the actual form factor computations.

II. HUBBARD LADDERS TO $SO(8)$ GROSS-NEVEU

Here we will briefly review the connection between Hubbard ladders (and related armchair carbon nanotubes) and the $SO(8)$ Gross-Neveu model developed in [1]. Specifically, we summarize the map between the two models, and interpret the excitation spectrum and fields of the $SO(8)$ Gross-Neveu model in terms of the original ladder model.

A. Map: D-Mott Hubbard Ladders to $SO(8)$ Gross-Neveu

The weakly interacting Hubbard ladder has five different phases, one massless and four massive [1]. The four massive phases, D-Mott (a spin liquid with approximate short-range d -wave pairing symmetry), S-Mott (a spin liquid with approximate short-range s -wave pairing symmetry), spin-Peierls (SP) (electrons are dimerized along the legs of the ladder), and a phase with charge density wave order (CDW), correspond to various combinations of attractive and repulsive interactions. We first focus upon the D-Mott phase which is characterized by generically repulsive interactions. Once we discuss the D-Mott phase in detail, we will return to the other three massive phases and demonstrate that each has a distinct $SO(8)$ symmetry [1]. In the massless phase, termed C2S2 (i.e. two charge bosons and two spin bosons), interactions are irrelevant, and so the phase has two trivial independent $SO(8)$ chiral symmetries. For this reason we will not consider it here.

We follow [1] in relating the D-Mott Hubbard ladder to an $SO(8)$ Gross-Neveu model. We begin with non-interacting electrons hopping on a ladder:

$$H_0 = - \sum_{x,\alpha} (ta_{1\alpha}^\dagger(x+1)a_{1\alpha}(x) + ta_{2\alpha}^\dagger(x+1)a_{2\alpha}(x) + t_\perp a_{1\alpha}^\dagger(x)a_{2\alpha}(x) + \text{h.c.}). \quad (2.1)$$

Here the a_l/a_l^\dagger are the electron annihilation/creation operators for the electrons on rung l of the ladder, x is a discrete coordinate along the ladder, and $\alpha = \uparrow, \downarrow$ describes electron spin. t and t_\perp describe respectively hopping between and along the ladder's rung.

The first step in the map is to reexpress the a 's of H_0 in terms of bonding/anti-bonding pairs:

$$c_{j\alpha} = \frac{1}{\sqrt{2}}(a_{1\alpha} + (-1)^j a_{2\alpha}). \quad (2.2)$$

With this transformation, the Hamiltonian can be diagonalized in momentum space in terms of two bands. Working at half filling, particle-hole symmetry dictates that the Fermi velocities, v_{Fj} , of the two bands, $j = 1, 2$, are equal. As we are interested in the low energy behaviour of the theory, the $c_{j\alpha}$'s are linearized about the Fermi surface, k_{Fj} :

$$c_{j\alpha} \sim c_{Rj\alpha} e^{ik_{Fj}x} + c_{Lj\alpha} e^{-ik_{Fj}x}, \quad (2.3)$$

where L, R corresponding to the right and left moving modes about the Fermi surface. With this H_0 becomes,

$$H_0 = v_F \int dx \sum_{j\alpha} [c_{Rj\alpha}^\dagger i\partial_x c_{Rj\alpha} - c_{Lj\alpha}^\dagger i\partial_x c_{Lj\alpha}]. \quad (2.4)$$

The next step is to bosonize the c 's:

$$c_{Pj\alpha} = \kappa_{j\alpha} e^{i\phi_{Pj\alpha}}, \quad P = +, - = R, L. \quad (2.5)$$

Here $\kappa_{j\alpha}$ are Klein factors satisfying

$$\{\kappa_{j\alpha}, \kappa_{i\beta}\} = 2\delta_{ij}\delta_{\alpha\beta}. \quad (2.6)$$

In terms of these four Bose fields, four new Bose fields are defined (effectively separating charge and spin):

$$\begin{aligned} \phi_{P1} &= \frac{1}{2}(\phi_{P1\uparrow} + \phi_{P1\downarrow} + \phi_{P2\uparrow} + \phi_{P2\downarrow}); \\ \phi_{P2} &= \frac{1}{2}(\phi_{P1\uparrow} - \phi_{P1\downarrow} + \phi_{P2\uparrow} - \phi_{P2\downarrow}); \\ \phi_{P3} &= \frac{1}{2}(\phi_{P1\uparrow} - \phi_{P1\downarrow} - \phi_{P2\uparrow} + \phi_{P2\downarrow}); \\ \phi_{P4} &= \frac{P}{2}(\phi_{P1\uparrow} + \phi_{P1\downarrow} - \phi_{P2\uparrow} - \phi_{P2\downarrow}). \end{aligned} \quad (2.7)$$

Note that ϕ_{P4} has a relative sign between the right and left movers. This sign effectively masks the $SO(8)$ symmetry of the original Hamiltonian. Without the sign change, 2.7 is no more than a triality transformation (see Section 2.4). If we then refermionize with these new bosons, i.e.,

$$\begin{aligned} \Psi_{Pa} &= \kappa_a e^{i\phi_a}, \quad a = 1, \dots, 3 \\ \Psi_{P4} &= P\kappa_4 e^{i\phi_4}, \end{aligned} \quad (2.8)$$

where the Klein factors are given by

$$\kappa_1 = \kappa_{2\uparrow}, \quad \kappa_2 = \kappa_{1\uparrow}, \quad \kappa_3 = \kappa_{1\downarrow}, \quad \kappa_4 = \kappa_{2\downarrow}, \quad (2.9)$$

we find for the free Hamiltonian,

$$H = \int dx \sum_a [\Psi_{aL}^\dagger i\partial_x \Psi_{aL} - \Psi_{aR}^\dagger i\partial_x \Psi_{aR}], \quad (2.10)$$

where the Fermi velocity, v_F , has been set to 1.

The point to the bosonization, change of basis, and refermionization only becomes apparent when one considers interactions. The discovery in [1] was that if one writes down a generic set of weak, left-right, repulsive interactions between electrons on the rungs, expresses these interactions in terms of the refermionized fermions, and then allows the couplings to flow under an RG, one finds that the interacting Hamiltonian is attracted to,

$$H_{int} = g \left[\sum_{a=1}^4 (i\Psi_{aL}^\dagger \Psi_{aR} - i\Psi_{aR}^\dagger \Psi_{aL}) \right]^2. \quad (2.11)$$

This is, of course, H_{int} for the $SO(8)$ Gross-Neveu model.

It will sometimes prove convenient to recast the theory in terms of Majorana fermions, ψ_{aP} . In terms of the Dirac fermions, Ψ_{aP} , they are given by

$$\Psi_{aP} = \frac{1}{\sqrt{2}} (\psi_{2a,P} + i\psi_{2a-1,P}), \quad (a = 1, \dots, 4). \quad (2.12)$$

In this basis, H_{int} can be recast as

$$H_{int} = g G_R^{ab} G_L^{ab}, \quad (a > b = 1, \dots, 8), \quad (2.13)$$

where $G_P^{ab} = i\psi_{aP}\psi_{bP}$ is one of the 28 $SO(8)$ Gross-Neveu currents.

B. D-Mott Excitations in Gross-Neveu

The Gross-Neveu $SO(8)$ model has an exceedingly rich spectrum. There are 24 fermionic particles of mass m organized into one eight dimensional vector representation and two eight dimensional spinor representations. We denote the particles of the vector representation by A_a , $a = 1, \dots, 8$. The A_a 's are the Majorana fermions of 2.12. The kink particles, in turn, will be denoted by A_α . Here α is of the form $\alpha = (\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$ and so takes on 16 values. These 16 particles decompose into the two eight-dimensional spinor representations. This is discussed in more detail in Section 4.

Beyond the eight dimensional representations, there are 29 bosonic particle states of mass $\sqrt{3}m$, transforming as a rank-two tensor of dimension 28 and a singlet. Together they form a representation of the $SO(8)$ Yangian symmetry. These particles can be thought of as bound states of either two kinks or two fundamental fermions.

As $SO(8)$ is a rank 4 algebra, the $SO(8)$ Gross-Neveu model has four Cartan bosons (i.e. the ϕ_{Pa} , $a = 1, \dots, 4$) and so its excitations are characterized by four quantum numbers, N_i , $i = 1, \dots, 4$. With the Majorana fermions, the combination

$$A_{2a} \pm iA_{2a-1}, \quad (2.14)$$

carries quantum number $N_a = \pm 1$, $N_b = 0$, $b \neq a$. The quantum numbers carried by the kinks A_α are directly encoded in α . If $\alpha = (a_1, a_2, a_3, a_4)$, $a_i = \pm 1/2$, the A_α carries the quantum numbers, $N_i = a_i$. The quantum numbers carried by the rank two tensor states can be directly deduced from the particles forming the bound state. As we will always think of the bound states in this way, we will not list their quantum numbers directly.

The last thing needed in the section is to identify the relationship between the quantum numbers, N_i , and the physical quantum numbers of the system, the z-component of spin, S_z , the charge, Q , the difference in z-component of spin between the two bands, S_{12} , and the "relative band chirality", P_{12} , defined as $P_{12} = N_{R1} - N_{L1} - N_{R2} + N_{L2}$, where N_{Pj} is the number electrons in band j with chirality P . In [1] it was found:

$$\begin{aligned} (N_1 = 1, 0, 0, 0) &\leftrightarrow (Q = 2, S_z = 0, S_{12} = 0, P_{12} = 0); \\ (0, N_2 = 1, 0, 0) &\leftrightarrow (Q = 0, S_z = 1, S_{12} = 0, P_{12} = 0); \\ (0, 0, N_3 = 1, 0) &\leftrightarrow (Q = 0, S_z = 0, S_{12} = 1, P_{12} = 0); \\ (0, 0, 0, N_4 = 1) &\leftrightarrow (Q = 0, S_z = 0, S_{12} = 0, P_{12} = 2). \end{aligned} \quad (2.15)$$

With this assignment, we can see that the vector representation of fundamental fermions corresponds to states of two electrons in the original formulation. For example, the fermion $A_2 \pm iA_1$ carries charge ± 2 and no spin (the cooperons), and the fermion $A_4 \pm iA_3$ carries spin, $S_z = 1$, and no charge (the magnons). This makes concrete the earlier comment that the change of basis of bosons in 2.7 affects a charge-spin separation. The spinor representations, the kinks, in turn correspond to single particle excitations as their quantum numbers are combinations of $N_i/2$.

C. D-Mott Gross-Neveu Fields

In this section we make contact between the fields of the $SO(8)$ Gross-Neveu model and the original fields of the Hubbard ladders. As we have already discussed, the fundamental (Dirac) fermions of the vector representation are given by

$$\begin{aligned} \Psi_{aP} &= \kappa_a e^{i\phi_{aP}}, \\ \Psi_{aP} &= P\kappa_a e^{i\phi_{aP}}, \end{aligned} \quad (2.16)$$

and carry quantum numbers corresponding to two electronic excitations. However the Ψ_{aP} are fermionic, whereas such excitations are bosonic. As such, Ψ_{aP} are

not simply related to a fermionic bilinear of the original electrons but must be a fermionic bilinear multiplying some non-local field (a Jordan-Wigner string). As we will not compute correlators involving such fields in this paper, we will not elaborate upon this (see [1]).

As discussed previously, the kinks correspond to single particle excitations. Thus we expect to find that the kink fields are related to the original electron operators. This is true in part. There are 32 kinks in total (counting both left and right movers), but only sixteen electron operators, the c 's and c^\dagger 's (four for each of the four Fermi points). So we expect only 1/2 of the kinks to correspond to actual electron operators.

We represented the fundamental fermions in terms of the four Cartan bosons. There is a corresponding representation for the kink fields:

$$\psi_{\alpha P} \sim e^{i\alpha \cdot \bar{\phi}_P}, \quad (2.17)$$

where $\bar{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$. The kink fields that then correspond to the electron operators c 's are as follows:

$$\begin{aligned} c_{R1\uparrow} &\sim e^{i(\phi_{1R} + \phi_{2R} + \phi_{3R} + \phi_{4R})/2}, \\ c_{R2\uparrow} &\sim e^{i(\phi_{1R} + \phi_{2R} - \phi_{3R} - \phi_{4R})/2}, \\ c_{R2\downarrow} &\sim e^{i(\phi_{1R} - \phi_{2R} + \phi_{3R} - \phi_{4R})/2}, \\ c_{R1\downarrow} &\sim e^{i(\phi_{1R} - \phi_{2R} - \phi_{3R} + \phi_{4R})/2}, \end{aligned}$$

(even chirality)

$$\begin{aligned} c_{L1\uparrow} &\sim e^{i(\phi_{1L} + \phi_{2L} + \phi_{3L} - \phi_{4L})/2}, \\ c_{L2\uparrow} &\sim e^{i(\phi_{1L} + \phi_{2L} - \phi_{3L} + \phi_{4L})/2}, \\ c_{L2\downarrow} &\sim e^{i(\phi_{1L} - \phi_{2L} + \phi_{3L} + \phi_{4L})/2}, \\ c_{L1\downarrow} &\sim e^{i(\phi_{1L} - \phi_{2L} - \phi_{3L} - \phi_{4L})/2}, \end{aligned}$$

(odd chirality). (2.18)

With hermitian conjugates, this totals to sixteen fields. The \sim sign is meant to indicate that these equivalences hold up to Klein factors. The $c_{Pj\alpha}$'s, of course, are fermionic. However the kink fields as defined are not.

The last set of fields that are of concern to us are the currents. The electric current of the ladder has the lattice representation

$$J \sim -i \sum_{l\alpha} \left[a_{l\alpha}^\dagger(x) a_{l\alpha}(x+1) - a_{l\alpha}^\dagger(x+1) a_{l\alpha}(x) \right], \quad (2.19)$$

where we have summed over the contribution coming from each spin (α) and each leg (l) of the ladder. Taking the continuum limit, J equals, in Gross-Neveu language,

$$J \sim i \sin k_{F1} \partial_t \phi_1 \sim G_{12}, \quad (2.20)$$

where G_{12} is one of the $SO(8)$ currents discussed in 2.13.

D. Other Massive Phases

By adjusting the signs of interaction couplings, three other massive phases, S-Mott, SP, and CDW, can be obtained [1]. Each of these phases is characterized by a distinct $SO(8)$ symmetry. However they share a common $SO(5)$ subgroup. The $SO(5)$ symmetry is the one-dimensional analog of the $SO(5)$ symmetry recently proposed by S.C. Zhang as a means to unify antiferromagnetism and superconductivity. Each of these new $SO(8)$ symmetries is readily expressible in terms of the D-Mott $SO(8)$ symmetry through considering the four Cartan bosons Φ_{Pa} [1]. As the first two Cartan generators belong to the $SO(5)$ subgroup they, however, remain unchanged. It is the latter two bosons that are affected. For completeness we review each of new phases in turn.

S-Mott:

The defining four Cartan bosons of the S-Mott phase, ϕ_{Pa}^S , are related to those of the D-Mott phase via

$$\begin{aligned} \phi_{Pa}^S &= \phi_{Pa}, \quad a = 1, 2, 3 \\ \phi_{P4}^S &= \phi_{P4} + P\pi/2. \end{aligned} \quad (2.21)$$

The sole difference between the $SO(8)$ algebra of the S-Mott phase and that of the D-Mott phase is a shift of ϕ_{Pa} by $P\pi/2$. This shift forces a sign change in the pair field correlator, changing the symmetry from d -wave to s -wave. The shift, however, does not change the excitation spectrum or its attendant assignment of quantum numbers, nor does it change field assignments beyond a phase.

Spin-Peierls (SP):

The defining four Cartan bosons of the SP phase, ϕ_{Pa}^{SP} , are related to those of the D-Mott phase via

$$\begin{aligned} \phi_{Pa}^{SP} &= \phi_{Pa}, \quad a = 1, 2, 4 \\ \phi_{P3}^{SP} &= P\phi_{P3}; \end{aligned} \quad (2.22)$$

that is, the sole change is to flip the sign on ϕ_{L3} . The effect of this sign change is two-fold. The quantum number associated with the third Cartan boson is now defined to be

$$\begin{aligned} N_3 = S_{12} &= \frac{1}{2}(N_{R\uparrow 1} - N_{R\downarrow 1} - N_{R\uparrow 2} + N_{R\downarrow 2}) - \\ &\quad \frac{1}{2}(N_{L\uparrow 1} - N_{L\downarrow 1} - N_{L\uparrow 2} + N_{L\downarrow 2}), \end{aligned} \quad (2.23)$$

where $N_{Pj\sigma}$ is the number of electron of chirality P in band j with spin σ , and so N_3 is the relative right-moving spin between the two bands minus the relative

left-moving spin between the two bands. Moreover the left moving electrons, c_L , are now to be identified with kinks of even chirality as opposed to the odd chirality kinks for the D-Mott phase (see 2.18).

Charge Density Wave (CDW):

The defining four Cartan bosons of the CDW phase, ϕ_{Pa}^{CDW} , are given by

$$\begin{aligned}\phi_{Pa}^{CDW} &= \phi_{Pa}, \quad a = 1, 2 \\ \phi_{P3}^{CDW} &= P\phi_{P3} \\ \phi_{P4}^{CDW} &= \phi_{P4} + P\pi/2.\end{aligned}\quad (2.24)$$

Because of the sign change of ϕ_{P3} , we again, as with the SP phase, have a redefinition of N_3 and reassociation of the kinks with the physical electrons. The shift of ϕ_{P4} , as with the S-Mott phase, only leads to phase multiplication of certain fields.

E. Triality

Of the $SO(2N)$ groups, $SO(8)$ possesses a special symmetry called triality that rotates its three fundamental representations (the vector and the two spinors) among one another. It is unique to $SO(8)$ as only in this case do the spinor and vector representations have the same dimension. The symmetry is isomorphic to Z_3 (hence its name) and so has a generator, g , such that $g^3 = 1$.

To exhibit the action of g upon the three representations, we consider its behaviour on the four Cartan bosons, ϕ_{aP} , introduced in 2.7. Under g the ϕ_i 's are linearly transformed as follows

$$\begin{aligned}\phi_{1P} &\rightarrow (\phi_{1P} + \phi_{2P} + \phi_{3P} + \phi_{4P})/2; \\ \phi_{2P} &\rightarrow (\phi_{1P} + \phi_{2P} - \phi_{3P} - \phi_{4P})/2; \\ \phi_{3P} &\rightarrow (\phi_{1P} - \phi_{2P} + \phi_{3P} - \phi_{4P})/2; \\ \phi_{4P} &\rightarrow (\phi_{1P} - \phi_{2P} - \phi_{3P} + \phi_{4P})/2.\end{aligned}\quad (2.25)$$

Given that the four Cartan bosons, ϕ_{aP} are identified with the four quantum numbers, N_i , g acts to redefine them correspondingly.

As the fermions are represented by $e^{i\phi_{aP}}$ and the kinks by $e^{i\alpha\cdot\phi_P}$, we can see the triality transformation acts to take the fermions to the kinks of even chirality (i.e. the number of components of α that are negative is even), the even kinks to the odd kinks, and the odd kinks to the fermions.

We will use triality to fix some of the properties of the form factors in Section 5. Via triality, we can relate form factors involving fermionic and kink fields and excitations, thus constraining them.

III. PRESENTATION OF RESULTS

In this section we present the computations using form factors of the optical conductivity, σ , the single particle spectral function, $A(k, w)$, and a tunneling $I - V$ curve. These results are generic to any of the four massive $SO(8)$ phases of the Hubbard ladders/carbon nanotubes. The optical conductivity is the same in each of the phases as they share a common $SO(5)$ subalgebra that contains the conserved electric charge. The computation of the spectral function and tunneling $I - V$ curve is done with respect to generic kinks. There is no need to specify which exact kink one is working with as one obtains identical results independent of the kink type (predominantly because of the action of the $SO(8)$ symmetry). Thus the results are independent of the identification of the kinks with the physical electronic excitations one makes in each of the massive phases.

In the Sections 4 and 5 that follow, we lay out the calculation of the form factors. For those uninterested in the details, the bulk of these sections may be skipped with only the results at the end of Section 5 referenced.

A. Optical Conductivity

In this section we consider the response of the ladder system to an electric field polarized along the legs. Apart from the treatment in [1], this problem has been examined previously, both theoretically [34] and experimentally [35]. However these two latter papers did not consider undoped ladders at zero temperature.

In linear response, the optical conductivity is given by

$$\text{Re}[\sigma(\omega, k)] = \text{Im}\left[\frac{\Delta(\omega, k)}{\omega}\right], \quad (3.1)$$

where Δ is the current-current correlator

$$\Delta(\omega, k) = \int dx d\tau e^{i\omega\tau} e^{ikx} \langle T(J(x, \tau)J(0, 0)) \rangle|_{\omega \rightarrow -i\omega + \delta}. \quad (3.2)$$

J is given by 2.20,

$$J \sim G_1^{12}. \quad (3.3)$$

To compute the correlator, $\langle T(G_1^{12}(x, \tau)G_1^{12}(0, 0)) \rangle$, we insert a resolution of the identity between the two J 's, turning the correlator into a form factor sum. We then have

$$\begin{aligned} \langle T(G_1^{12}(x, \tau) G_1^{12}(0, 0)) \rangle &= \sum_{n=0}^{\infty} \sum_{a_1, \dots, a_n} \int \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi} \langle G_1^{12}(0) | A_{a_1}^\dagger(\theta_1) \dots A_{a_n}^\dagger(\theta_n) \rangle \\ &\times \langle A_{a_n}(\theta_n) \dots A_{a_1}(\theta_1) | G_1^{12}(0) \rangle \exp\left(-|\tau| \sum_{i=1}^n m_{a_i} \cosh(\theta_i) + ix \sum_{i=1}^n m_{a_i} \sinh(\theta_i)\right), \end{aligned} \quad (3.4)$$

where the first sum \sum_n runs over the number of particles in the form factor expansion and the second sum \sum_{a_i} runs over the different particle types. We have also extracted the spacetime dependence of each term.

To compute this sum in its entirety is generally an intractable problem. The usual solution is to truncate the sum at some n . Here we will content ourselves with a truncation at the two particle level:

$$\begin{aligned} \langle T(G_1^{12}(x, \tau) G_1^{12}(0, 0)) \rangle &= \int \frac{d\theta_1}{2\pi} \langle G_1^{12}(0) | A_{12}^\dagger(\theta_1) \rangle \langle A_{12}(\theta_1) | G_1^{12}(0) \rangle \\ &\times \exp\left(-|\tau| \sqrt{3}m \cosh(\theta_1) + ix \sqrt{3}m \sinh(\theta_1)\right) \\ &+ \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \exp\left(-|\tau| m(\cosh(\theta_1) + \cosh(\theta_2)) + ix m(\sinh(\theta_1) + \sinh(\theta_2))\right) \\ &\times \left(\sum_{ab} \langle G_1^{12}(0) | A_a^\dagger(\theta_1) A_b^\dagger(\theta_2) \rangle \langle A_b(\theta_2) A_a(\theta_1) | G_1^{12}(0) \rangle \right. \\ &\left. + \sum_{\alpha\beta} \langle G_1^{12}(0) | A_\alpha^\dagger(\theta_1) A_\beta^\dagger(\theta_2) \rangle \langle A_\beta(\theta_2) A_\alpha(\theta_1) | G_1^{12}(0) \rangle \right). \end{aligned} \quad (3.5)$$

The first term gives the single particle contribution to the correlation function. The only particle that contributes here is A_{12} , denoting one of the particles belonging to the rank 2 tensor multiplet. At the two particle level a variety of contributions are non-zero. The second term in 3.5 gives the contribution of two Majorana fermions while the third term gives the contribution of kinks with the same chirality.

As indicated in the introduction, this truncation of the form factor sum is better than it may at first seem. Because the correlator is evaluated at zero temperature in a massive system, the higher order terms make contributions only at higher energies, ω . That is, the massiveness of the system leads to particle thresholds. The next contribution comes from a three particle combination of even kink/fermion/odd kink that carries mass $3m$. Thus for

$\omega < 3m$, this term gives no contribution to $\text{Re}[\sigma(\omega, k)]$, for arbitrary k . Hence our result for $\text{Re}[\sigma(\omega, k)]$, is exact for $\omega < 3m$.

In the case when ω does exceed $3m$, we expect the higher particle form factors to make only a small contribution to $\sigma(\omega)$. In practice, form factor sums have been found to be strongly convergent for operators in massive theories [23, 26–28]. To obtain a good approximation to correlators involving such fields, only the first few terms need to be kept. Even in massless theories where there are no explicit thresholds, convergence is good provided the engineering dimension of the operator matches its anomalous dimension [24, 25].

Using the results for the form factors of Section 5, we can put everything together and write down an expression for $\text{Re}[\sigma(\omega)]$:

$$\begin{aligned} \text{Re}[\sigma(\omega)] &= \delta(\omega - \sqrt{3m^2 + k^2}) \left(\frac{A_G}{m} \right)^2 \frac{2}{9} \sqrt{\frac{\pi}{3}} \frac{\Gamma(1/6)}{\Gamma(2/3)} \exp\left[-2 \int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} s^2(x/3)\right] \\ &+ \theta(\omega^2 - k^2 - 4m^2) \frac{12m^2 A_G^2}{(\omega^2 - k^2 - 3m^2)^2} \frac{\omega \sqrt{\omega^2 - k^2 - 4m^2}}{(\omega^2 - k^2)^{3/2}} \\ &\times \exp\left[\int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} (1 - c(x) \cos(\frac{x\theta_{12}}{\pi}))\right], \end{aligned} \quad (3.6)$$

where $s(x) = \sinh(x)$, $c(x) = \cosh(x)$, and

$$\theta_{12} = \cosh^{-1} \left[\frac{\omega^2 - k^2 - 2m^2}{2m^2} \right]. \quad (3.7)$$

As indicated in Section 5, A_G is an arbitrary constant normalizing all current form-factors while $G_c(x)$ can be found in (5.68).

In Figure 1 we plot the real part of the optical conductivity for wavevector $k = 0$. There is no Drude weight as we are at zero temperature and zero doping. We see there is an exciton type peak at $\omega = \sqrt{3}m$ corresponding to the single particle form factor contribution. The first vertical dashed line marks out the beginning of the two particle form factor contribution to the conductivity. The onset of the two particle contribution behaves as $\sqrt{\omega - 2m}$ and not as $1/\sqrt{\omega - 2m}$ as would be expected in a free theory due to the divergence in the density of states, the van-Hove singularity, that generically occurs in one dimensional systems. This singularity is removed by the corresponding current matrix element which behaves as $(\omega - 2m)$ with $\omega \rightarrow 2m^+$ as is expected generically because the low energy behaviour becomes strongly renormalized in the presence of even weak interactions [29].

The optical conductivity was computed in [1] using the large N limit of $SO(2N)$, or in an alternate language, an RPA approximation. In such an approximation, the model becomes equivalent to a theory of four massive, non-interacting Dirac fermions. Hence [1] finds that the van-Hove singularity is present.

The second vertical dashed line in Figure 1 at $3m$ marks the point where the three particle form factors would begin to make a contribution. Up to this point, the result is exact. We note that the three particle contribution is strictly a consequence of interactions. In $SO(8)$ language, two kinks of opposite chirality together with a fermion will couple to the current operator. In a free theory these different particles would not all exist and there would be no three particle contribution.

If we were to compute the three particle contribution, three possibilities present themselves. The three particle density of states approaches a constant as $\omega \rightarrow 3m^+$. If the corresponding matrix element vanishes as $\omega \rightarrow 3m^+$, the contribution opens up gradually, leaving $\sigma(\omega)$ continuous at $\omega = 3m$. If the three particle matrix element also approaches a constant value as $\omega \rightarrow 3m^+$, the conductivity is marked by a jump at $\omega = 3m$. But if the matrix element diverges in this limit, we expect a corresponding divergence in the conductivity at $\omega = 3m$.

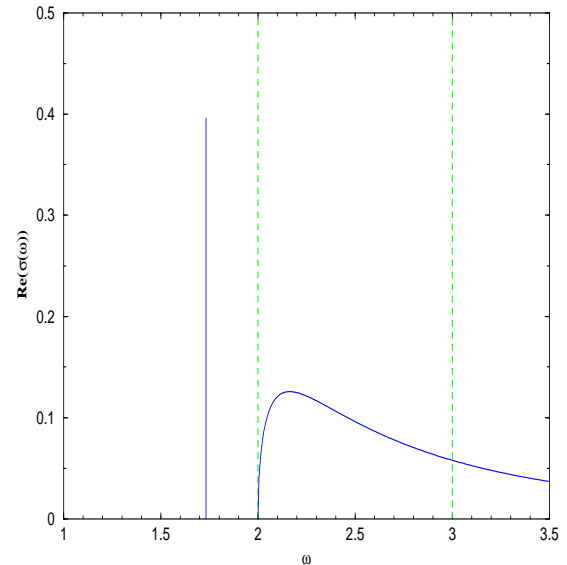


Figure 1: Plot of the optical conductivity at wavevector $k = 0$.

In Figure 2 we plot the real part of the optical conductivity for wavevector $k = \pi/3$. As a result of the form of $\text{Re}[\langle \sigma \rangle]$ (see 3.6), shifting k from 0 to $\pi/3$ shifts the single particle form factor contribution to $m\sqrt{\pi^2/9 + 3}$ and the onset of the two particle contribution to $m\sqrt{\pi^2/9 + 4}$. We see that as k grows, the contribution from the single particle moves closer to the two particle threshold.

It can now be asked how perturbations to $SO(8)$ Gross-Neveu will affect the computation of the optical conductivity. We consider this in the broadest terms by focusing upon how the spectrum of $SO(8)$ Gross-Neveu is changed under a perturbing term. We do so through straightforward stationary perturbation theory, in the same spirit that [21] treated the off-critical Ising model in a magnetic field. The most general possible perturbation takes the form

$$H_{\text{pert}} = \lambda G_{ab} G_{cd}, \quad (3.8)$$

where G_{ab} , G_{cd} are $SO(8)$ currents (of unspecified chirality). For such a perturbation it is necessary to consider degenerate perturbation theory. Thus in a given particle multiplet (for example, the fundamental fermions in the vector representation), the perturbed energies arise through diagonalizing the matrix

$$M_{ij} = \langle A_i^\dagger(\theta) H_{\text{pert}} A_j(\theta) \rangle, \quad (3.9)$$

where here the index i, j indicates the particles A_i, A_j belong to the multiplet of concern. In the case that $G_{ab} = G_{cd}$, M_{ij} is necessarily diagonal, i.e. nondegenerate perturbation theory is sufficient.

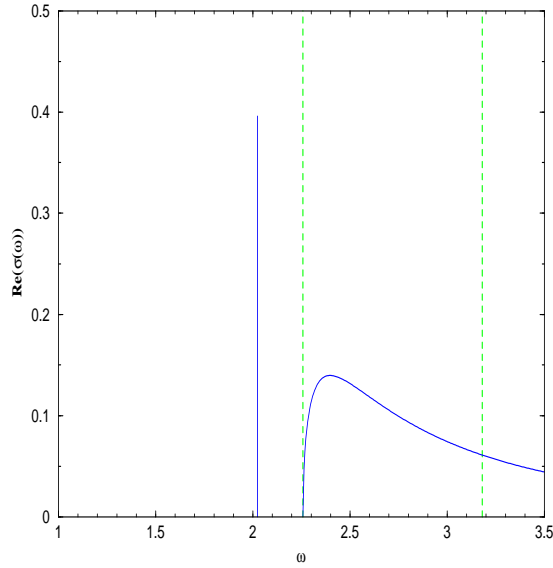


Figure 2: Plot of the optical conductivity at wavevector $k = \pi/3$. This plot is exact up to $w/m = \sqrt{\pi^2/9 + 9}$ at which point the three particle states begin to make a contribution.

It is important to emphasize that this procedure can be handled in the context of integrability. The expression in 3.9 is no more than a form-factor which can readily be computed. Moreover as the theory is massive, perturbation theory is well controlled. We expect the unperturbed theory to describe all qualitative features of the model while the perturbations to only introduce small quantitative changes.

The consequences of a perturbation for the optical conductivity are two-fold. We expect the exciton peak (found, for example, in Figure 1 at $k = 0$ and $\omega = \sqrt{3}$) to split. In the unperturbed model the peak results from a single rank two bosonic bound state coupling to the current operator. When the matrix M_{ab} above is diagonalized, this particular state should be mixed into many others resulting in several states that couple to the current operator. However we do not expect the functional forms of the exciton peaks to change: they should remain delta functions. They must do so provided the perturbation is not so large as to push the exciton peak past the threshold of two particle states where it then conceivably could decay. As there is a gap between the excitonic peak and the two particle threshold, this will not happen for small perturbations. Experimentally changes to the exciton peak may not be detectable. Given that any experiment will be conducted at finite temperature, the excitonic peak will be thermally broadened, perhaps washing out any splitting of the original zero temperature peak.

We also expect the perturbation to affect the onset of the two particle threshold, although in a less dramatic

fashion. Like the unperturbed case, there will be several two particle contributions to the optical conductivity. However unlike the unperturbed case, the thresholds of the two particle contributions will not all occur at $\omega = 2m$ but be distributed about this energy. Thus the two particle contribution is arrived at (approximately) by superimposing several slightly shifted two-particle contributions similar to those found in Figures 1 and 2. But given the optical conductivity vanishes at threshold, the qualitative picture remains effectively unchanged (i.e. the superimposed contributions will appear nearly identical to the original picture). That the optical conductivity vanishes at the two particle threshold is a result of the vanishing of the relevant matrix element at threshold. This should be robust under perturbation as it is ultimately a consequence of the mere presence of interactions and not some particular type of interactions.

B. Single Particle Spectral Function

In this section we compute the single-particle spectral function of the electrons of the Hubbard ladder. To do so we first consider the correlator,

$$G(k_x, k_y, \tau) = \sum_{l=1,2} \int_{-\infty}^{\infty} dx e^{-ik_y l - ik_x x} \times \langle T(a_{l\alpha}(x, \tau) a_{l\alpha}^\dagger(0, 0)) \rangle. \quad (3.10)$$

Here k_y takes on the values $0, \pi$. We then define the particle/hole spectral functions, $A_{p/h}$, as follows:

$$A_p(\mathbf{k}, \omega) + A_h(-\mathbf{k}, -\omega) = \text{Im} \int_{-\infty}^{\infty} d\tau e^{-i\omega\tau} G(k_x, k_y, \tau) \Big|_{\omega \rightarrow -i\omega + \delta}. \quad (3.11)$$

In keeping with [1], we have not explicitly summed over spin, α .

As described in Section 2, electronic excitations around the Fermi point correspond in the Gross-Neveu language to low energy excitations of kinks. We thus expect to recast the Greens function, G , above in terms of kink correlators. This in fact can be done with the result,

$$G(Pk_{Fi} + k, k_{yi}, \tau) = \int_{-\infty}^{\infty} dx e^{ikx} \langle T(c_{Pi\alpha}(x, \tau) c_{Pi\alpha}^\dagger(0, 0)) \rangle, \quad (3.12)$$

where $i = 1, 2$ and $k_{yi} = (2 - i)\pi$. The c 's, the bonding-anti-bonding electrons given in 2.2, are in turn related to the various kinks via 2.18. The Greens function on the r.h.s. of 3.12 is thus equal to

$$\langle T(c_{Pi\alpha}(x, \tau) c_{Pi\alpha}^\dagger(0, 0)) \rangle = \langle T(\kappa_\alpha \psi_\pm^\alpha(x, \tau) \kappa_{\bar{\alpha}} \psi_\pm^{\bar{\alpha}}(0, 0)) \rangle. \quad (3.13)$$

where α ($\bar{\alpha}$ being its charge conjugate) is the particular kink corresponding to the Fermi point (k_{Fi}, k_{yi}) . The κ_α are Klein factors included to ensure the ψ^α are anti-commuting. Because of the $SO(8)$ symmetry together with its associated triality symmetry, $\langle T(\kappa_\alpha \psi_\pm^\alpha(x, \tau) \kappa_{\bar{\alpha}} \psi_\pm^{\bar{\alpha}}(0, 0)) \rangle$ turns out to be independent of the type α of kink. It is only sensitive to whether the kink field is right (+) or left (-) moving.

To compute this correlator, we again expand to the two lowest contributions:

$$\begin{aligned} \langle T(\kappa_\alpha \psi_\pm^\alpha(x, \tau > 0) \kappa_{\bar{\alpha}} \psi_\pm^{\bar{\alpha}}(0, 0)) \rangle = \\ \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \langle \psi_\pm^\alpha(x, \tau) A_{\bar{\alpha}}^\dagger(\theta_1) \rangle \langle A_{\bar{\alpha}}(\theta_1) \psi_\pm^{\bar{\alpha}}(0) \rangle \\ + \sum_{a\beta} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \langle \psi_\pm^\alpha(x, \tau) A_\beta^\dagger(\theta_2) A_a^\dagger(\theta_1) \rangle \end{aligned}$$

$$\times \langle A_a(\theta_1) A_\beta(\theta_2) \psi_\pm^{\bar{\alpha}}(0) \rangle. \quad (3.14)$$

The first contribution, the one particle contribution, comes from the kink excitation, $A_{\bar{\alpha}}$, destroyed by the field, ψ_α . The second contribution, a two particle contribution, arises from kinks, A_β , of opposite chirality to A_α , and Majorana fermions, A_a . (This reflects the group theoretical fact that the tensor product of an $SO(8)$ spinor representation with an $SO(8)$ vector representation gives the other $SO(8)$ spinor representation [36].) The first contribution not included is a bound state-kink pair. It begins to contribute at $\omega = (1 + \sqrt{3})m$.

From the form factor expressions from Section 5, we can then write down the expression for the spectral functions, $A_{p/h}(\omega, k)$,

$$\begin{aligned} A_p(\omega, Pk_{Fi} + k, k_{yi}) = A_h(\omega, -Pk_{Fi} + k, k_{yi}) = \frac{\pi |c_P|^2}{m} \frac{\omega + Pk}{\sqrt{k^2 + m^2}} \delta(\omega - \sqrt{k^2 + m^2}) \\ + \theta(\omega - \sqrt{k^2 + 4m^2}) \frac{16m^4 A_F^2}{\omega - Pk} \frac{1}{(\omega^2 - k^2 - m^2)^2} \frac{1}{\sqrt{\omega^2 - k^2 - 4m^2}} \\ \times \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} (1 - c(x) \cos(\frac{x\theta_{12}}{\pi})) \right], \end{aligned} \quad (3.15)$$

where A_F is the (unspecified) normalization of the two particle kink form-factor, $G_f(x)$ is given in (5.70), and c_\pm is found in (5.73). For $P = R = +$ (i.e. right-moving electrons/kinks), this function is plotted in Figure 3.

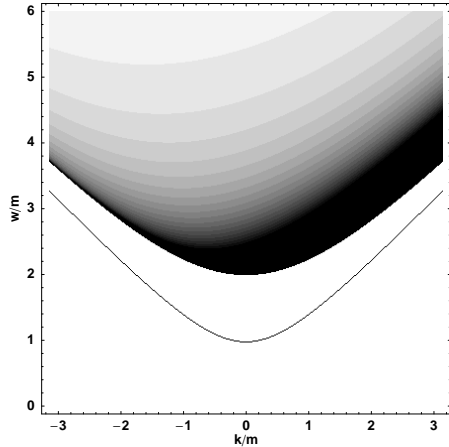


Figure 3: Plot of the single particle spectral function for right moving kinks. The more darkly shaded region corresponds to greater spectral weight.

The parabolic line in Figure 3 arises from the single particle form factor contribution, and represents the standard dispersion relation of a particle of mass, m . Above this curve comes the two particle form factor contribution to the spectral function. This contribution is bounded by the curve, $\omega = \sqrt{k^2 + 4m^2}$, and so the single particle states do not cross into the two particle region. As can be seen from 3.15, the two particle contribution opens up at threshold with a square-root singularity, indicative of the van-Hove singularity in the density of states.

The plot is manifestly chiral with weighting greater for $k > 0$ than for $k < 0$. This is to be expected as we are plotting the excitations linearized about the Fermi momentum, $+k_{Fi}$. The heavier weighting for $k > 0$ indicates that is easier to create excitations above the Fermi sea than below it. It is interesting indeed that excitations below the Fermi surface can be created at all and is a mark that interactions are at play.

As 3.15 indicates, we have computed the single particle spectral function in the neighborhood of the Fermi points. However in [1], the authors found two particle spectral weight to exist away from the Fermi points at harmonics $k = \pm 3k_{Fi}$ and $\omega = 2m$, that is the same energy threshold as at the Fermi points, k_{Fi} , themselves. They identified the spectral weight at these points through combining the $SO(8)$ picture of the ladders with the original lattice formulation. As such, we are unable

to say anything definitive about spectral weight at these points as we are forced to remain with the $SO(8)$ description.

The effect of a perturbation breaking $SO(8)$ can be sketched in a similar fashion to that of the optical conductivity. In the case of the single particle contribution to the spectral function, we expect that under degenerate perturbation theory the single parabolic line in Figure 3 to split akin to the behaviour of the excitonic peak in the optical conductivity. Again this effect will be masked by finite temperatures which thermally broaden such features.

At the two particle level, both in the unperturbed and perturbed cases, several sets of two-particle pairs contribute. In the unperturbed case, these sets all begin to contribute at $\omega = \sqrt{4m^2 + k^2}$. In the perturbed case, the sets will begin to contribute at different points. For small shifts, a plot such as that in Figure 3 will remain much the same. However it is worthwhile to point out that with each two-particle pair is associated a square-root divergence due to the van-Hove singularity in the density of states. Thus we would expect to find a series of such singularities in the presence of a perturbation.

C. Tunneling Current

In this section we study the tunneling between a metallic lead and the carbon nanotube/Hubbard ladder through a point contact. Our starting point is a Lagrangian describing the nanotube/ladder, the metallic lead, and the tunneling interaction:

$$\mathcal{L} = \mathcal{L}_{SO(8)} + \mathcal{L}_{lead} + \mathcal{L}_{tun}. \quad (3.16)$$

$\mathcal{L}_{SO(8)}$ is the Lagrangian of the $SO(8)$ Gross-Neveu model.

The electron gas in the lead is, in general, three dimensional. However, in the context of tunneling through a point contact, the electron gas can be mapped onto an one dimensional chiral fermion (see for example [37–39]). The general idea is well illustrated by its application to the Kondo problem. There an electron scatters off a spin impurity at $x = 0$. The scattering is determined by the electron operator, $\psi(x = 0)$. As $\psi(0)$ only depends on its spherically symmetric, $L = 0$, mode, one can consider the scattering electron in terms of an ingoing and outgoing radial model defined on the half-line, $r \in [0, \infty]$. Unfolding the system onto the full line leaves one with a chiral fermion. We emphasize however that the map requires no special symmetry; the result is exact regardless of particular anisotropies [39]. As a consequence, we write \mathcal{L}_{lead} as

$$\mathcal{L}_{lead} = \frac{1}{8\pi} \Psi^\dagger \partial_z \Psi, \quad (3.17)$$

where Ψ is a massless, left moving fermion, and $z = (\tau + ix)/2$.

It remains to specify \mathcal{L}_{tun} . In order to preserve charge, the electrons must couple to the kinks of the $SO(8)$ Gross-Neveu model, the excitations with the quantum numbers of the electron. Thus

$$\mathcal{L}_{tun} = g_L [\Psi^\dagger(\tau) \psi_-^\alpha(\tau) + \psi_-^{\bar{\alpha}}(\tau) \Psi(\tau)] \delta(x) + g_R [\Psi^\dagger(\tau) \psi_+^\alpha(\tau) + \psi_+^{\bar{\alpha}}(\tau) \Psi(\tau)] \delta(x). \quad (3.18)$$

Here we have coupled the lead electrons to both the right and left moving fields creating the kink, α , and have allowed the two couplings, g_L and g_R , to be unequal. However as we will work to lowest non-vanishing order in the tunneling matrix elements $g_{L/R}$, the tunneling current will depend upon the sum, $g_L^2 + g_R^2$, that is, the contribution of the left and right channels to tunneling will add linearly. Similarly, permitting other kinks to couple to the lead electrons will give lowest order contributions which simply add.

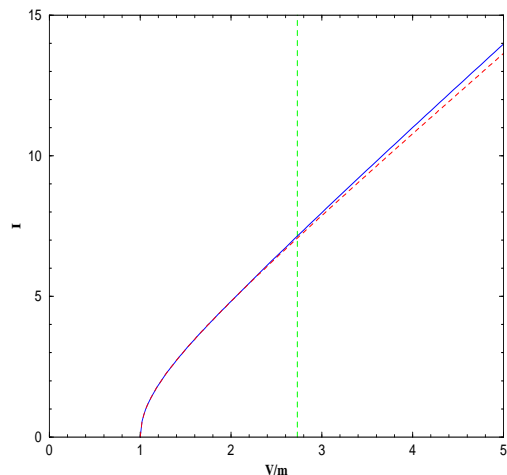


Figure 4: Plot of the tunneling current as a function of applied voltage. The dashed curve describes tunneling into a non-interacting fermionic system of mass, m . The vertical dashed line marks where a second set of two particle states begins to make a contribution.

The calculation of the current to lowest non-vanishing order in $g_{L/R}$ follows the standard route. The tunneling current operator is given by

$$I(\tau) = ig_R (\psi_+^{\bar{\alpha}}(\tau) \Psi(\tau) - \Psi^\dagger(\tau) \psi_+^\alpha(\tau)) + ig_L (\psi_-^{\bar{\alpha}}(\tau) \Psi(\tau) - \Psi^\dagger(\tau) \psi_-^\alpha(\tau)). \quad (3.19)$$

In order to induce current flow, $\langle I(\tau) \rangle$, through the point contact, one biases the lead with a voltage, V . This bias can be taken into account via a gauge transformation,

$$\Psi(\tau) \rightarrow e^{iV\tau} \Psi(\tau). \quad (3.20)$$

In effect we have shifted the energy levels of the electrons. Treating the couplings, g_L/g_R , with linear response theory, we find

$$\begin{aligned} \langle I(\omega) \rangle = g_L^2 \text{Re} \Big\{ & \int d\tau i e^{i\omega\tau} [\\ & e^{-iV\tau} \langle \psi_-^\alpha(\tau) \psi_-^{\bar{\alpha}}(0) \rangle \langle \Psi^\dagger(\tau) \Psi(0) \rangle \\ & - e^{iV\tau} \langle \psi_-^{\bar{\alpha}}(\tau) \psi_-^\alpha(0) \rangle \langle \Psi(\tau) \Psi^\dagger(0) \rangle] \Big|_{\substack{\omega \rightarrow -i\omega + \delta \\ V \rightarrow -iV}} \Big\} \\ & + (L \rightarrow R, \psi_-^\alpha/\psi_-^{\bar{\alpha}} \rightarrow \psi_+^\alpha/\psi_+^{\bar{\alpha}}). \end{aligned} \quad (3.21)$$

The lead electron correlator $\langle \Psi^\dagger(\tau) \Psi(0) \rangle$ is well known:

$$\langle \Psi^\dagger(\tau, x) \Psi(0) \rangle = \langle \Psi(\tau, x) \Psi^\dagger(0) \rangle = \frac{1}{\tau + ix}. \quad (3.22)$$

With this it is straightforward to express the dc current, $\langle I(\omega = 0) \rangle$, in terms of the single particle kink spectral function,

$$\langle I(\omega = 0) \rangle = \frac{1}{2\pi} \int_{-V}^V d\omega \int_{-\infty}^{\infty} dk [g_L^2 A_-(\omega, k) + g_R^2 A_+(\omega, k)], \quad (3.23)$$

where $A_\pm(\omega, k) = A_p(\omega, \pm k_{Fi} + k, k_{yi}) + A_h(-\omega, \mp k_{Fi} - k, k_{yi})$, and $A_{p/h}$ are the spectral functions given in 3.15. We note that as a technical point, in deriving the above equation we have displaced, Ψ , the lead electron operator, slightly from $x = 0$. In this way we cure the UV divergence attendant as $\tau \rightarrow 0$. At the end of the calculation we then take x to 0.

In the previous section we have computed $A_\pm(\omega, k)$ exactly for energies $\omega < (\sqrt{3} + 1)m$. Inserting 2.15 into 3.23, we find $\langle I(0) \rangle$ takes the form,

$$\begin{aligned} \langle I(0) \rangle = & \frac{|c_\pm|^2}{m} (g_R^2 + g_L^2) (V^2 - m^2)^{1/2} \theta(|V| - m) \text{sgn}(V) \\ & + \theta(|V| - 2m) \text{sgn}(V) \times \text{two particle contribution}. \end{aligned} \quad (3.24)$$

We see that for $|V| < 2m$, the system behaves as a gapped free fermion. The first sign that there is any interaction comes for $|V| > 2m$ where the voltage begins to probe the two particle states, a signature of interacting fermions.

We explicitly plot $\langle I(0) \rangle$ in Figure 4. The square root behavior near $V/m = 1$ and subsequent linear form is

typical of a gapped fermion. At $V/m = 2$, the two particle states begin to contribute leading to a small change in the slope of the $I - V$ curve. At $V/m = \sqrt{3} + 1$ (marked by the vertical dashed line), a second set of two particle states (a mass $\sqrt{3}m$ bound state together with a kink) begin to contribute and at this point the result ceases to be exact. However as with the current correlators, we expect this higher energy contribution to be small.

The change in slope in the $I - V$ curve at $V/m = 2$ can be explicitly computed. To do so we consider $\partial_V \langle I(0) \rangle$. This quantity is given by

$$\partial_V \langle I(0) \rangle = \frac{1}{\pi} (g_R^2 + g_L^2) \int_{-\infty}^{\infty} dk A_\pm(V, k). \quad (3.25)$$

We can thus see $\partial_V \langle I(0) \rangle$ directly measures the local density of states at $x = 0$ of the nanotube/ladder system.

We plot $\partial_V \langle I(0) \rangle$ in Figure 5. The square root singularity at $V = m$ signals the singularity of the density of states in an one dimensional system. At $V = 2m$ we see a sudden jump, indicative of the onset of the two particle contribution. The height of the jump can be determined exactly:

$$\begin{aligned} \partial_V \langle I(0) \rangle (V/m = 2^+) - \partial_V \langle I(0) \rangle (V/m = 2^-) = \\ \frac{8A_F^2}{9m} (g_R^2 + g_L^2) \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} (1 - c(x)) \right]. \end{aligned} \quad (3.26)$$

The region $m < V < 2m$ of $\partial_V \langle I(0) \rangle$ completely determines m (by the location of the jump), as well as an overall scale (the product of $(g_L^2 + g_R^2)$ and the constant A_F , normalizing the spectral function). Dividing out these non-universal numbers leaves a universal number, characterizing the magnitude of the jump:

$$\frac{8}{9} \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} (1 - c(x)) \right]. \quad (3.27)$$

This number represents a definite prediction based upon the integrability of the model.

We now consider the approximate effect of perturbations breaking integrability on the tunneling conductance. As the tunneling conductance is determined directly from the single particle spectral function, we can deduce how the former is affected from how the latter is changed. At a given momentum, the single particle contribution to the single particle spectral function under perturbation comes at a discrete set of energies. In terms of the tunneling conductance, we expect a series of closely spaced square root divergences (a sawtooth behaviour) about $V = m$ indicative of a series of van-Hove singularities. As the perturbation is removed these singularities would collapse on top of one another leaving the original picture in Figure 5.

In the unperturbed case the two-particle threshold is characterized by a jump in the differential conductance.

Under a perturbation, this jump would become a staircase or a series of smaller, closely spaced jumps. This is a reflection of the series of van-Hove singularities found about $\omega = \sqrt{k^2 + 4m^2}$ in the two particle contribution to the single particle spectral function.

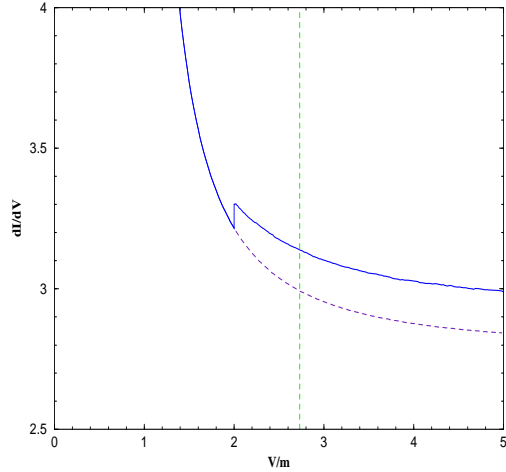


Figure 5: Plot of the differential conductance as a function of applied voltage. The dashed curve marks the single particle contribution to this quantity while the solid curve gives both the single and first two particle contribution. The latter plot is exact up to $V/m = \sqrt{3} + 1$ (indicated by the dashed vertical line) where a bound state-kink pair begins to make a contribution.

IV. S-MATRICES FOR $SO(8)$ GROSS-NEVEU

In this section we review the S-matrices for scattering among the kinks and elementary (Majorana) fermions. From these we derive S-matrices for scattering between Dirac fermions, kinks and Dirac fermions, and rank 2 bound states and Dirac fermions. Knowledge of these S-matrices will allow us to compute the various form factors needed to obtain the results in the previous section. Before beginning it is necessary to lay out the group theory of $SO(8)$. In particular it is necessary to describe the spinor representations (i.e. the kinks) of $SO(8)$.

A. $SO(8)$ Spinor Representations

The spinor representations are expressed in terms of $SO(8)$ γ -matrices. The γ -matrices in turn are built out of the two-dimensional Pauli matrices, σ^i 's. As we are interested in $SO(8)$, we consider four copies of the σ^i 's,

$$\sigma_1^i, \sigma_2^i, \sigma_3^i, \sigma_4^i, \quad (4.1)$$

each acting on a different two-dimensional space:

$$\sigma_n^i |\alpha_1, \dots, \alpha_4\rangle = |\alpha_1, \dots, \sigma_n^i \alpha_n, \dots, \alpha_4\rangle. \quad (4.2)$$

A particular basis vector, $|\alpha\rangle$, in the corresponding 16 dimensional vector space can be labeled by a series of four $\pm 1/2$, i.e.

$$\alpha = |\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2\rangle. \quad (4.3)$$

Physically, the kink associated with $|\alpha\rangle$ then carries $\pm 1/2$ of the four $U(1)$ quantum numbers, N_i , corresponding to the Cartan elements of $SO(8)$, discussed above.

In terms of the σ^i 's, the γ -matrices are defined by (following the conventions of [40]),

$$\begin{aligned} \gamma_{2n-1} &= \sigma_n^1 \otimes \prod_{j=1}^{n-1} \sigma_j^3; \\ \gamma_{2n} &= \sigma_n^2 \otimes \prod_{j=1}^{n-1} \sigma_j^3; \end{aligned} \quad (4.4)$$

for $1 \leq n \leq 4$. These matrices satisfy the necessary Clifford algebra:

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}. \quad (4.5)$$

In this representation the Clifford-algebra generators, γ_n , are imaginary and anti-symmetric for n even, while for n odd, they are real and symmetric. The $SO(8)$ -generators are represented by

$$\sigma^{ab} = \frac{1}{2}(\gamma^a \gamma^b - \gamma^b \gamma^a), \quad (4.6)$$

in analogy with the more familiar $SO(4)$ case.

The 16-dimensional space of the γ 's decomposes into two 8-dimensional spaces, each of which forms one of the two irreducible $SO(8)$ spinor representations. The decomposition is achieved explicitly via the hermitian chirality operator, Γ ,

$$\Gamma = \prod_{a=1}^8 \gamma^a. \quad (4.7)$$

Γ is such that it commutes with all $SO(8)$ generators and is diagonal with eigenvalues ± 1 . If $|\alpha^\pm\rangle$ is a state with an even (odd) number of negative components (i.e. states with positive or negative isotopic chirality), then

$$\Gamma|\alpha^\pm\rangle = \pm|\alpha^\pm\rangle. \quad (4.8)$$

Thus the operators, $(1 \pm \Gamma)/2$, project onto the two irreducible subspaces.

The last item to be presented in this section is the charge conjugation matrix, C . In terms of the γ 's, C is given by

$$C = \gamma_2 \gamma_4 \gamma_6 \gamma_8. \quad (4.9)$$

C is completely off-diagonal (as expected). If $|\bar{\alpha}\rangle$ is the anti-particle of a kink, $|\alpha\rangle$, then

$$|\bar{\alpha}\rangle = C_{\alpha\beta} |\beta\rangle. \quad (4.10)$$

C is such that $C^2 = 1$, $C^T = C$, and $C\gamma^a C^{-1} = \gamma^{a*}$.

B. S-Matrices for the Elementary Fermions and Kinks

To describe scattering we introduce Faddeev-Zamolodchikov operators, $A_a(\theta)$ and $A_\alpha(\theta)$, that create the elementary fermions and kinks respectively. θ is the rapidity which parameterizes a particle's energy and momentum:

$$p = m \sinh(\theta); \quad E = m \cosh(\theta). \quad (4.11)$$

Because the $SO(8)$ Gross-Neveu model is integrable, scattering is completely encoded in the two-body S-matrix. This S-matrix, in turn, is encoded in the commutation relations of the Faddeev-Zamolodchikov operators:

$$A_1(\theta_1)A_2(\theta_2) = S_{12}^{34}(\theta_{12})A_4(\theta_2)A_3(\theta_1). \quad (4.12)$$

$S_{12}^{34}(\theta_{12})$ is the amplitude of a process by which particles 1 and 2 (be they kinks, fundamental fermions, or bosonic bound states) scatter into 3 and 4. It is a function of $\theta_{12} \equiv \theta_1 - \theta_2$ by reason of Lorentz invariance.

Scattering between the fundamental Majorana fermions, first determined in [41], is given by

$$S_{ab}^{cd}(\theta) = \delta_{ab}\delta_{cd}\sigma_1(\theta) + \delta_{ac}\delta_{bd}\sigma_2(\theta) + \delta_{ad}\delta_{bc}\sigma_3(\theta), \quad (4.13)$$

where the σ 's are defined by

$$\begin{aligned} \sigma_1(\theta) &= -\frac{i}{3} \frac{\pi}{i\pi - \theta} \sigma_2(\theta); \\ \sigma_3(\theta) &= -i \frac{\pi}{3\theta} \sigma_2(\theta); \\ \sigma_2(\theta) &= -Q(\theta)Q(i\pi - \theta) \frac{s(\theta) + i\sqrt{3}/2}{s(\theta) - i\sqrt{3}/2}. \end{aligned} \quad (4.14)$$

where $s(\theta) \equiv \sinh(\theta)$. Q is given to be

$$Q(\theta) = \frac{\Gamma(1/6 - i\theta/2\pi)\Gamma(1/2 - i\theta/2\pi)}{\Gamma(-i\theta/2\pi)\Gamma(2/3 - i\theta/2\pi)}. \quad (4.15)$$

$S_{ab}^{cd}(\theta)$ satisfies a Yang-Baxter equation,

$$S_{a_1 a_2}^{c_1 c_2}(\theta_{12}) S_{c_1 c_3}^{b_1 b_3}(\theta_{13}) S_{c_2 c_3}^{b_2 b_3}(\theta_{23}) =$$

$$S_{a_2 a_3}^{c_2 c_3}(\theta_{23}) S_{a_1 c_3}^{c_1 b_3}(\theta_{13}) S_{c_1 c_2}^{b_1 b_2}(\theta_{12}). \quad (4.16)$$

Physically, the Yang-Baxter equation encodes the equivalence of different ways of representing three-body interactions in terms of two-body amplitudes. Formally, it expresses the associativity of the Faddeev-Zamolodchikov algebra. The S-matrix, $S_{ab}^{cd}(\theta)$, also satisfies both a crossing,

$$S_{ab}^{cd}(\theta) = S_{bc}^{da}(i\pi - \theta), \quad (4.17)$$

and unitarity relation,

$$S_{ab}^{cd}(\theta) S_{cd}^{ef}(-\theta) = \delta_a^e \delta_b^f. \quad (4.18)$$

These constraints determine S_{ab}^{cd} up to a 'CDD'-factor. Such factors determine the pole structure in the physical strip, $\text{Re}(\theta) = 0$, $0 < \text{Im}\theta < 2\pi$, of the scattering matrix and so are indicative of bound states. Here the CDD factor is

$$\frac{s(\theta) + i\sqrt{3}/2}{s(\theta) - i\sqrt{3}/2}, \quad (4.19)$$

and reflects the fact that two fermions can form a bound state of mass $\sqrt{3}m$ transforming as a component of the 29 dimensional representation of the $SO(8)$ Yangian. As stated in the introduction, this representation is formed out of a rank-2 anti-symmetric $SO(8)$ tensor and a scalar. The overall sign of the S-matrix is determined by examining the residue of the pole in S_{aa}^{aa} at $\theta = i\pi/3$. This pole is indicative of the formation of a mass $\sqrt{3}m$ scalar bound state in the s-channel and so should have positive imaginary residue.

Scattering between the kinks and fundamental fermions is given by

$$S_{a\alpha}^{b\beta} = -t_1(\theta)\delta_{ab}\delta_{\alpha\beta} - t_2(\theta)\sigma_{\beta\alpha}^{ba}. \quad (4.20)$$

The form of 4.18, first written down in [42], is determined by writing down all possible $SO(8)$ invariant tensors with indices a, b and α, β . t_1 and t_2 are described by

$$\begin{aligned} t_1(\theta) - t_2(\theta) &= \frac{\theta - i2\pi/3}{\theta + i2\pi/3} S_2(\theta); \\ t_1(\theta) + 7t_2(\theta) &= S_2(\theta), \end{aligned} \quad (4.21)$$

where $S_2(\theta)$ is given by

$$S_2(\theta) = \frac{s(\theta) + i\sqrt{3}/2}{s(\theta) - i\sqrt{3}/2} \frac{\Gamma(5/6 + i\theta/2\pi)\Gamma(4/3 - i\theta/2\pi)}{\Gamma(5/6 - i\theta/2\pi)\Gamma(4/3 + i\theta/2\pi)}. \quad (4.22)$$

Again, the Yang-Baxter equation, crossing, and unitarity are used to determine 4.21 and 4.22. These constraints were first explicitly solved in [40]. The overall sign is

fixed by insisting the pole in $S_{\alpha\alpha}^{a\alpha}$ at $\theta = i2\pi/3$, indicative of a kink formed in the s-channel as a fermion-kink bound state, has positive imaginary residue.

Kink-kink scattering takes the form

$$S_{\alpha\beta}^{\gamma\delta} = \frac{1}{16} \sum_{r=0}^8 \frac{u_r(\theta)}{r!} \sigma_{\gamma\beta}^{(r)} \sigma_{\delta\alpha}^{(r)}, \quad (4.23)$$

where $\sigma^{(r)}$ is a rank- r anti-symmetric tensor:

$$\sigma^{(r)} \equiv \sigma^{a_1 \dots a_r} = (\gamma^{a_1} \dots \gamma^{a_r})_A. \quad (4.24)$$

Here A represents a complete antisymmetrization of the gamma matrices. By $\sigma_{\gamma\beta}^{(r)} \sigma_{\delta\alpha}^{(r)}$ we mean a trace over all possible rank- r anti-symmetric tensors:

$$\sigma_{\gamma\beta}^{(r)} \sigma_{\delta\alpha}^{(r)} = \sum_{a_1 \dots a_r} \sigma_{\gamma\beta}^{a_1 \dots a_r} \sigma_{\delta\alpha}^{a_1 \dots a_r}. \quad (4.25)$$

Again the generic form of $S_{\alpha\beta}^{\gamma\delta}$ was determined in [42], while the specific forms of the u 's were given in [40]. There it was found

$$\begin{aligned} u_{4+r}(\theta) &= (-1)^r u_{4-r}(\theta); \\ u_{r+2}(\theta) &= u_r(\theta) \frac{(1-r/3) - (1+i\theta/\pi)}{(1-r/3) + (1+i\theta/\pi)}; \\ u_2(\theta) &= \sigma_2(i\pi - \theta) - \sigma_3(i\pi - \theta); \\ u_1(\theta) &= S_2(i\pi - \theta). \end{aligned} \quad (4.26)$$

The $SO(8)$ Gross-Neveu model has an isotopic chirality conservation law [40]. Thus opposite chirality kink scattering is determined solely by u_n , n odd, while same chirality kink scattering is determined solely by u_n , n even. The overall sign of the S-matrix again is chosen so that appropriate s-channel poles have positive imaginary residues. That the u 's bear a close relationship to amplitudes from fermion-fermion scattering and kink-fermion scattering is a reflection of the triality symmetry of $SO(8)$.

C. S-Matrices for Dirac Fermions

On occasion the Majorana fermion basis will not be the most convenient. For example, as these fermions do not carry definite U(1) charge, they will not couple nicely to a chemical potential introduced when the system is doped. Thus we would want to consider, as in [6], a fermion basis with well-defined U(1) charges, i.e. we would want to reexpress the Majorana fermions as Dirac-fermions.

As there are eight Majorana fermions in $SO(8)$ Gross-Neveu, we have four Dirac fermions. Denoting the corresponding Faddeev-Zamolodchikov operators by $D_i^\pm(\theta)$, where $D_i^-(\theta)$ is $D_i^+(\theta)$'s anti-particle, we have

$$D_i^\pm(\theta) = \frac{1}{\sqrt{2}} (A_{2i}(\theta) \pm i A_{2i-1}(\theta)) \quad 1 \leq i \leq 4, \quad (4.27)$$

where the A_i 's are Majorana fermions. Because scattering is determined by the commutation relations,

$$D_i^{c_i}(\theta_1) D_j^{c_j}(\theta_2) = S_{(ic_i)(jc_j)}^{(kc_k)(lc_l)} D_k^{c_k}(\theta_2) D_l^{c_l}(\theta_1), \quad (4.28)$$

where $c = \pm 1$ is indicative of the U(1) charge carried, we can determine $S_{(ic_i)(jc_j)}^{(kc_k)(lc_l)}$ from 4.13. We find the following,

$$\begin{aligned} S_{(ic_i)(jc_j)}^{(kc_k)(lc_l)}(\theta) &= \delta_{ij} \delta_{kl} C_D^{c_i c_j} C_D^{c_k c_l} \sigma_1(\theta) \\ &\quad + \delta_{ik} \delta_{c_i c_k} \delta_{jl} \delta_{c_l c_j} \sigma_2(\theta) \\ &\quad + \delta_{il} \delta_{c_i c_l} \delta_{jk} \delta_{c_k c_j} \sigma_3(\theta), \end{aligned} \quad (4.29)$$

where $C_D \equiv \sigma_1$ is the charge conjugation matrix for Dirac fermions. We point out that two Dirac fermions of the same U(1) charge will scatter diagonally, i.e.

$$D_i^\pm(\theta_1) D_i^\pm(\theta_2) = (\sigma_2(\theta_{12}) + \sigma_3(\theta_{12})) D_i^\pm(\theta_2) D_i^\pm(\theta_1). \quad (4.30)$$

We will also want to consider scattering between kinks and Dirac fermions. From 4.20 we find

$$\begin{aligned} S_{(ic_i)\alpha}^{(jc_j)\beta}(\theta) &= \delta_{ij} \delta_{c_i c_j} \delta_{\alpha\beta} (t_1(\theta) - t_2(\theta)) \\ &\quad + \frac{c_i c_j}{2} (\Gamma_{c_j}^j \Gamma_{-c_i}^i)_{\beta\alpha} t_2(\theta). \end{aligned} \quad (4.31)$$

where

$$\Gamma_\pm^j = \gamma^{2j-1} \pm i \gamma^{2j}. \quad (4.32)$$

When $A_\alpha(\theta)$ carries U(1) charge $c_i/2$ (i.e. when the U(1) charges of $A_\alpha(\theta)$ and D_i carry the same sign) the above form simplifies greatly:

$$S_{(ic_i)\alpha}^{(ic_i)\beta}(\theta) = t_1(\theta) - t_2(\theta), \quad (4.33)$$

i.e. scattering becomes diagonal. This is exploited in [6] to compute the excitation energy of a kink in the doped system.

We want to consider one last S-matrix involving the Dirac fermions, that of a Dirac fermion with rank 2 tensorial bound states. We will not do this in general. Rather we consider a bound state carrying two non-zero positive charges, say $U_i = +1$ and $U_j = +1$, scattering off a Dirac fermion carrying charge either $U_i = +1$ or $U_j = +1$. This is precisely the situation encountered when treating spin 1 excitations in the doped system.

We thus introduce the bound state Faddeev-Zamolodchikov operator, $A_B(\theta)$. $A_B(\theta)$ can be thought of as a bound state of two Dirac fermions with charge

$U_i = +1$ and $U_j = +1$, and consequently can be represented in terms of the corresponding two Faddeev-Zamolodchikov operators,

$$ig_{(i+)(j+)}^B A_B(\theta) = \text{res}_{\delta=0} A_{i+}(\theta + \delta + i\bar{u}_{(i+)\bar{B}}^{(j-)}) A_{j+}(\theta - i\bar{u}_{(j+)\bar{B}}^{(i-)}), \quad (4.34)$$

where $\text{res}_{\delta=0}$ denotes the residue at $\delta = 0$ and \bar{B} is the charge conjugate of B . In writing 4.34, we have taken the particle normalization to be $\langle \theta | \theta' \rangle = 2\pi\delta(\theta - \theta')$. The \bar{u} 's are given by

$$\bar{u}_{sB}^{s'} = \pi - u_{sB}^{s'}, \quad (4.35)$$

where $iu_{sb}^{s'}$ is the location of the pole indicative of the particle s' in the s - B S-matrix. Here we have

$$u_{(i+)\bar{B}}^{(j-)} = u_{(j+)\bar{B}}^{(i-)} = 5\pi/6. \quad (4.36)$$

The g 's are related to the residues of the poles in the $(i+)-(j+)$ S-matrix and can be interpreted as the amplitude to form the bound state from $(i+)$ and $(j+)$. If $u_{(i+)(j+)}^B$ is the location of the pole indicative of B , $g_{(i+)(j+)}^B$ is defined by

$$S_{(i+)(j+)}^{(i+)(j+)}(\theta) \sim i \frac{g_{(i+)(j+)}^B g_B^{(i+)(j+)}}{\theta - iu_{(i+)(j+)}^B}. \quad (4.37)$$

It is then easy to show (from 4.30) that

$$\begin{aligned} g_{(i+)(j+)}^B g_B^{(i+)(j+)} &= S_0; \\ g_{(j+)(i+)}^B g_B^{(i+)(j+)} &= -S_0; \end{aligned} \quad (4.38)$$

where S_0 is some constant. Hence $g_{(j+)(i+)}^B / g_{(i+)(j+)}^B = -1$. This last relation is all we will need to determine the scattering between $A_B(\theta)$ and $D_{i,j}^+$.

From 4.32 and 4.28 we find that $S_{B(i/j+)}^{B(i/j+)}$ is given by

$$\begin{aligned} S_{B(i/j+)}^{B(i/j+)}(\theta) &= \sigma_2(\theta - i\pi/6)(\sigma_2(\theta + i\pi/6) + \sigma_3(\theta + i\pi/6)) \\ &\quad - \sigma_3(\theta - i\pi/6)\sigma_3(\theta + i\pi/6) \\ &= \frac{\theta + i\frac{\pi}{6}}{\theta - i\frac{\pi}{6}} (\sigma_2(\theta + i\pi/6) + \sigma_3(\theta + i\pi/6)) \\ &\quad \times (\sigma_2(\theta - i\pi/6) + \sigma_3(\theta - i\pi/6)), \end{aligned} \quad (4.39)$$

so that scattering between $A_B(\theta)$ and $D_{i,j}^+$ is diagonal.

V. FORM FACTORS

Here we determine the needed form factors to compute correlators in the undoped system. If the reader is uninterested in the actual derivation, the results are summarized at the end of the section.

A. Basic Properties: Two Particle Form Factors

The two particle form factors of a field $\psi(x)$ are defined as the matrix elements,

$$f_{12}^\psi(\theta_1, \theta_2) = \langle \psi(0) A_2(\theta_2) A_1(\theta_1) \rangle. \quad (5.1)$$

The form of $f_{12}^\psi(\theta_1, \theta_2)$ is constrained by integrability, braiding relations, Lorentz invariance, and hermiticity.

The constraint coming from integrability arises from the scattering of Faddeev-Zamolodchikov operators. For 5.1 to be consistent with 4.12, we must have

$$f_{21}^\psi(\theta_2, \theta_1) = S_{12}^{34}(\theta_{12}) f_{34}^\psi(\theta_1, \theta_2). \quad (5.2)$$

The second constraint can be thought of as a periodicity axiom. It reads

$$f_{21}^\psi(\theta_2, \theta_1) = R_{\psi_2\psi} f_{12}^\psi(\theta_1 - 2\pi i, \theta_2). \quad (5.3)$$

$R_{\psi_2\psi}$ is a phase⁸ that arises from the non-locality of the fields in $SO(8)$ Gross-Neveu. This non-locality leads to non-trivial braiding relations between fields which $R_{\psi_2\psi}$ encodes:

$$\psi_2(x, t) \psi(y, t) = R_{\psi_2\psi} \psi(y, t) \psi_2(x, t); \quad x < y. \quad (5.4)$$

ψ_2 here is the field that is associated with the particle, $A_2(\theta_2)$. 5.3 can be derived through crossing symmetry constraints (see, for example, [33, 32]).

The form factor, f_{12}^ψ , must also satisfy Lorentz invariance. If ψ has Lorentz spin, s , f_{12}^ψ will take the form (at least in the cases at hand),

$$f_{12}^\psi(\theta_1, \theta_2) = e^{s(\theta_1 + \theta_2)/2} f(\theta_{12}), \quad (5.5)$$

where by virtue of $\theta_{12} \equiv \theta_1 - \theta_2$, f is a Lorentz scalar.

The constraints 5.2, 5.3, and 5.5, do not uniquely specify f_{12}^ψ . It is easily seen that if f_{12}^ψ satisfies these axioms then so does $f_{12}^\psi R(\cosh(\theta_{12}))$, where $R(x)$ is some rational expression. The strategy is then as follows. One first

⁸In certain circumstances R is a matrix, thus marking out the difference between abelian and non-abelian braiding.

determines the minimal solution to the constraints, minimal in the sense that it has the minimum number of zeros and poles in the physical strip, $\text{Re}(\theta) = 0, 0 < \text{Im}(\theta) < 2\pi$. Then one adds poles according to the theory's bound state structure. If the S-matrix element scattering particles 1,2 has a pole at $\theta = iu$, then $f_{12}^\psi(\theta_1, \theta_2)$ has a pole at

$$\theta_2 = \theta_1 + iu. \quad (5.6)$$

Insisting that f_{12}^ψ has such poles and only such poles fixes $R(x)$ up to a constant.

The phase of this constant can be readily determined. Appealing to hermiticity gives us

$$\begin{aligned} \langle \psi(0) A_2(\theta_2) A_1(\theta_1) \rangle^* &= \langle A_1^\dagger(\theta_1) A_2^\dagger(\theta_2) \psi^\dagger(0) \rangle \\ &= \langle \psi^\dagger(0) A_1(\theta_1 - i\pi) A_2(\theta_2 - i\pi) \rangle, \end{aligned} \quad (5.7)$$

where the last line follows by crossing and so

$$f_{12}^\psi(\theta_1, \theta_2)^* = f_{21}^{\psi^\dagger}(\theta_2 - i\pi, \theta_1 - i\pi). \quad (5.8)$$

For the form factors we will examine, this will be enough to fix their overall phase.

B. Basic Properties: One Particle Form Factors

One particle form factors are in a sense trivial; Lorentz invariance completely determines their form. If $\psi(x)$ has Lorentz spin, s , then

$$f_1^\psi(\theta) = \langle \psi(0) A_1(\theta) \rangle = c e^{s\theta}, \quad (5.9)$$

where c is some constant. To determine c we use the theory's bound state structure.

In analogy with 4.34 we write

$$ig_{23}^1 A_1(\theta) = \text{res}_{\delta=0} A_2(\theta + \delta + i\bar{u}_{21}^3) A_3(\theta - i\bar{u}_{31}^2). \quad (5.10)$$

Then we have

$$ig_{23}^1 f_1^\psi(\theta) = \text{res}_{\delta=0} f_{32}^\psi(\theta - i\bar{u}_{31}^2, \theta + \delta + i\bar{u}_{21}^3). \quad (5.11)$$

Thus knowledge of the two particle form factors completely determines their one-particle counterparts.

C. Braiding of the Fields

In order to employ the periodicity axiom 5.3, we need to specify the braiding of the fields. In order to do this for the kink and fermion fields, we identify these fields with their corresponding excitations, A_α and A_a . The braiding of the fields is then encoded in the asymptotic limits of the corresponding S-matrices (see [32]). Precisely,

if ${}_P R_{12}$ is defined by (where $P = \pm$ denotes right/left movers)

$$\psi_{1P}(x, t) \psi_{2P}(y, t) = {}_P R_{12} \psi_{2P}(y, t) \psi_{1P}(x, t), \quad x < y, \quad (5.12)$$

then ${}_P R_{12}$ is given by

$${}_P R_{12} = (S_{12}^{12}(+\infty))^P. \quad (5.13)$$

In this way we find

$$\pm R_{\alpha\beta} = \begin{cases} -1, & \alpha, \beta \text{ are the same chirality} \\ \pm i, & \alpha, \beta \text{ are of opposite chirality;} \end{cases} \quad (5.14)$$

and

$$\pm R_{a\beta} = \pm i. \quad (5.15)$$

The braiding relationship among the kinks differ from what one finds by defining the kink fields as vertex operators,

$$\psi_{\alpha P} = e^{i\alpha \cdot \vec{\phi}}, \quad (5.16)$$

and using the easily derived braiding relations:

$$e^{ia\phi_P(x,t)} e^{ib\phi_P(y,t)} = e^{iP\pi ab} e^{ib\phi_P(y,t)} e^{ia\phi_P(x,t)}, \quad x > y. \quad (5.17)$$

Similarly, the correct braiding of kinks with the Majorana fermions is not obtained by considering the braiding of the corresponding vertex operators.

The only other fields that we concern ourselves with in this paper are the currents. However the current are local fields and so have trivial braiding with the kinks and the fermions.

D. Two Particle Form Factors for the $SO(8)$ Currents

There are two possible two particle form factors for the $SO(8)$ currents, one involving two fermions,

$${}_μ f_{cd}^{ab}(\theta_1, \theta_2) = \langle G_\mu^{ab}(0) A_d(\theta_2) A_c(\theta_1) \rangle, \quad (5.18)$$

and one involving two kinks,

$${}_μ f_{\alpha\beta}^{ab}(\theta_1, \theta_2) = \langle G_\mu^{ab}(0) A_\beta(\theta_2) A_\alpha(\theta_1) \rangle, \quad (5.19)$$

where the two kinks have the same chirality. By the triality symmetry of $SO(8)$, these two form factors will have the same functional form. Indeed we will use triality to set their relative normalization, crucial to the calculation of the conductivity in Section 3.

Calculation of ${}_μ f_{cd}^{ab}$:

To determine ${}_μ f_{cd}^{ab}$, one first fixes its group theoretical structure. Given that G_μ^{ab} is anti-symmetric and that

$A_d A_c$ must be anti-symmetric in d and c in order to couple to G_μ^{ab} [36], we must look for an invariant tensor anti-symmetric in the pairs a, b and c, d . The only choice is $\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}$, and so ${}_\mu f_{cd}^{ab}$ takes the form,

$${}_\mu f_{cd}^{ab}(\theta_1, \theta_2) = (\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})f_\mu(\theta_1, \theta_2). \quad (5.20)$$

Lorentz invariance demands

$$f_\mu(\theta_1, \theta_2) = (e^{(\theta_1+\theta_2)/2} - (-1)^\mu e^{-(\theta_1+\theta_2)/2})f(\theta_{12}). \quad (5.21)$$

Having so constrained the form of ${}_\mu f_{cd}^{ab}(\theta_1, \theta_2)$, conditions 5.2 and 5.3 tell us $f(\theta_{12})$ must satisfy

$$\begin{aligned} f(-\theta) &= -f(\theta)(\sigma_2(\theta) - \sigma_3(\theta)); \\ f(-\theta) &= f(\theta - 2\pi i). \end{aligned} \quad (5.22)$$

As the current is bosonic, the braiding here is trivial. Because $\sigma_2 - \sigma_3$ can be written in the form,

$$\begin{aligned} \sigma_2(\theta) - \sigma_3(\theta) &= \exp \left[\int_0^\infty \frac{dx}{x} s\left(\frac{x\theta}{i\pi}\right) G_c(x) \right], \\ G_c(x) &= 2 \frac{c(x/6) - s(x/6)e^{-2x/3}}{c(x/2)}, \end{aligned} \quad (5.23)$$

it is readily checked that

$$f(\theta) = s(\theta/2) \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta)\right) \right], \quad (5.24)$$

minimally satisfies the above condition. To add the bound state structure, we note that the two fermions c, d form a bound state of mass $\sqrt{3}m$. This implies a pole in the c - d S-matrix at $\theta = iu = i\pi/3$, and so a pole in $f(\theta_{12})$ at $\theta_2 = \theta_1 + i\pi/3$. Hence we multiply $f(\theta)$ by $1/(c(\theta) - 1/2)$, leading us to the form for ${}_\mu f_{cd}^{ab}(\theta_1, \theta_2)$:

$$\begin{aligned} {}_\mu f_{cd}^{ab}(\theta_1, \theta_2) &= iA_G(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \\ &\times (e^{(\theta_1+\theta_2)/2} - (-1)^\mu e^{-(\theta_1+\theta_2)/2}) \frac{s(\theta_{12}/2)}{c(\theta_{12}) - 1/2} \\ &\times \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta_{12})\right) \right], \end{aligned} \quad (5.25)$$

where iA_G is some normalization with mass dimension $[m]$.

The phase of A_G is determined through the hermiticity condition

$${}_\mu f_{cd}^{ab}(\theta_1, \theta_2)^* = {}_\mu f_{dc}^{ab}(\theta_2 - i\pi, \theta_1 - i\pi). \quad (5.26)$$

This implies that A_G is real.

Calculation of ${}_\mu f_{\alpha\beta}^{ab}$:

Again we begin by identifying the group theoretical structure of ${}_\mu f_{\alpha\beta}^{ab}$:

$${}_\mu f_{\alpha\beta}^{ab}(\theta_1, \theta_2) = (C\sigma^{ab})_{\alpha\beta} f_\mu(\theta_1, \theta_2), \quad (5.27)$$

where C is the charge conjugation matrix introduced in Section 4. $(C\sigma^{ab})_{\alpha\beta}$ is not only anti-symmetric in a, b but also in α, β as it must be if the kinks are to couple to G_μ^{ab} [36]. Now $(C\sigma^{ab})_{\alpha\beta}$ is not the only available invariant tensor. One also has $(\sigma^{ab}C)_{\alpha\beta}$, anti-symmetric combinations built up out of $\gamma^a C \gamma^b$ or $C \gamma^a C \gamma^b C$, or some combination of all three (but not σ^{ab} as this choice violates obvious U(1) conservation). Perhaps the most natural choice is to make ${}_\mu f_{\alpha\beta}^{ab}$ explicitly C-symmetric:

$${}_\mu f_{\alpha\beta}^{ab}(\theta_1, \theta_2) = ((C\sigma^{ab})_{\alpha\beta} + (\sigma^{ab}C)_{\alpha\beta}) f_\mu(\theta_1, \theta_2). \quad (5.28)$$

However this forces ${}_\mu f_{\alpha\beta}^{(2a-1)(2a)}$ to zero and this is not consistent with triality, i.e. all ${}_\mu f_{cd}^{(2a-1)(2a)}$ do not vanish. So we choose as above. Note that we could equally well have chosen $(\sigma^{ab}C)_{\alpha\beta}$. That our choice is not C-symmetric is not surprising as the theory is not trivially C-symmetric. With kink-fermion scattering, a C-transformation changes the sign of selected amplitudes (see 4.20).

As before Lorentz invariance demands

$$f_\mu(\theta_1, \theta_2) = (e^{(\theta_1+\theta_2)/2} - (-1)^\mu e^{-(\theta_1+\theta_2)/2})f(\theta_{12}). \quad (5.29)$$

Then by 5.2, $f(\theta_{12})$ must satisfy

$$\begin{aligned} f(-\theta) &= -u_2(\theta) \frac{i\pi + 3\theta}{4\pi i - 3\theta} \frac{i\pi - \theta}{\theta} f(\theta), \\ &= -(\sigma_2(\theta) - \sigma_3(\theta))f(\theta), \end{aligned} \quad (5.30)$$

where the last line follows from 4.26. That 5.30 has exactly the same form as 5.22 is a non-trivial consequence of triality. The periodicity axiom also takes the same form as before:

$$f(-\theta) = f(\theta - 2\pi i). \quad (5.31)$$

As the bound states of two fermions are identical to that of two same chirality kinks, the bound state structure of ${}_\mu f_{\alpha\beta}^{ab}$ is the same as ${}_\mu f_{cd}^{ab}$. Hence

$$\begin{aligned} {}_\mu f_{\alpha\beta}^{ab}(\theta_1, \theta_2) &= iB(C\sigma^{ab})_{\alpha\beta} \\ &\times (e^{(\theta_1+\theta_2)/2} - (-1)^\mu e^{-(\theta_1+\theta_2)/2}) \frac{s(\theta_{12}/2)}{c(\theta_{12}) - 1/2} \\ &\times \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta_{12})\right) \right]. \end{aligned} \quad (5.32)$$

What is left is to determine the relative value of B to A_G via triality. An overall normalization is not so interesting when computing two-point correlators as the physics so encoded is not universal.

To determine the normalization we focus upon a particular current component G_0^{12} . G_0^{12} is given by

$$G_0^{12} \propto \partial_x \phi_1, \quad (5.33)$$

where ϕ_1 is one of the four Cartan bosons. Under the triality transformation of 2.25, G_0^{12} is transformed into

$$G_0^{12} \rightarrow \frac{1}{2}(G_0^{12} + G_0^{34} + G_0^{56} + G_0^{78}). \quad (5.34)$$

Now focus upon the form factor $f_{\alpha\beta}^{12}$ where $|\alpha\rangle = (1/2, 1/2, 1/2, 1/2)$ and $|\beta\rangle = (-1/2, -1/2, -1/2, -1/2)$. Under a naive counting of quantum numbers, A_α and A_β are transformed into Dirac fermions:

$$\begin{aligned} A_\alpha &\rightarrow \frac{1}{\sqrt{2}}(A_2 + iA_1) \\ A_\beta &\rightarrow \frac{1}{\sqrt{2}}(A_2 - iA_1). \end{aligned} \quad (5.35)$$

Hence we have

$$f_{\alpha\beta}^{12} \rightarrow \frac{i}{2} f_{12}^{12}. \quad (5.36)$$

This fixes $B = A_G/2$.

E. One Particle Form Factors for $SO(8)$ Currents

Only the rank-2 tensorial states couple individually to the G_μ^{ab} 's. Denoting the Faddeev-Zamolodchikov operators for the bound states by $A_{\{cd\}}$ we have (as in 4.34)

$$i g_{\{cd\}}^{cd} A_{\{cd\}}(\theta) = \text{res}_{\delta=0} A_c(\theta + \delta + i\pi/6) A_d(\theta - i\pi/6), \quad (5.37)$$

where A_c, A_d are Majorana fermions. $g_{cd}^{\{cd\}}$, the amplitude to form the bound state $A_{\{cd\}}$ from A_c and A_d , is given by the residue in S_{cd}^{cd} at $iu_{cd}^{\{cd\}}$:

$$S_{cd}^{cd}(\theta) \sim i \frac{g_{cd}^{\{cd\}} g_{\{cd\}}^{cd}}{\theta - iu_{cd}^{\{cd\}}}, \quad (5.38)$$

and so is readily computed to be

$$g_{cd}^{\{cd\}} = (2\sqrt{3\pi} \frac{\Gamma(2/3)}{\Gamma(1/6)})^{1/2}. \quad (5.39)$$

Note that the value of $g_{cd}^{\{cd\}}$ given here is ambiguous up to a phase. However the phase can be pinned down through an appeal to hermiticity.

Using 5.37, we can write down

$$\begin{aligned} \mu f_{\{cd\}}^{ab}(\theta) &= iA_G(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) \\ &\times (e^\theta - (-1)^\mu e^{-\theta}) \frac{1}{\sqrt{3}} \left(2\sqrt{3\pi} \frac{\Gamma(2/3)}{\Gamma(1/6)} \right)^{-1/2} \\ &\times \exp \left[- \int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} s^2(x/3) \right]. \end{aligned} \quad (5.40)$$

The overall phase (up to a sign) has been fixed by the constraint, $\mu f_{\{cd\}}^{ab} = -\mu f_{\{cd\}}^{ab}$, arising from hermiticity. Although the overall sign is uncertain, the computation of correlators does not depend on its value.

F. Two Particle Form Factors for Gross-Neveu Fermion Operators

At the two particle level, two kinks of opposite chirality will couple to a right/left fermion field, ψ_\pm^a . This leads to form factors, $\pm f_{\alpha\beta}^a(\theta_1, \theta_2)$, defined by

$$\pm f_{\alpha\beta}^a(\theta_1, \theta_2) = \langle \psi_\pm^a(0) A_\beta(\theta_2) A_\alpha(\theta_1) \rangle. \quad (5.41)$$

Covariance under $SO(8)$ demands $\pm f_{\alpha\beta}^a$ takes the form

$$\pm f_{\alpha\beta}^a(\theta_1, \theta_2) = [c_1(C\gamma^a)_{\alpha\beta} + c_2(\gamma^a C)_{\alpha\beta}] f_\pm(\theta_1, \theta_2), \quad (5.42)$$

where c_1 and c_2 are constants. (That c_1 and c_2 are not more generally independent functions of θ_1 and θ_2 is easily seen; the constraints 5.2 and 5.3 do not allow it.) c_1 and c_2 can be fixed easily. Suppose $|\alpha\rangle = (1/2, 1/2, 1/2, 1/2)$ and $|\beta\rangle = (1/2, -1/2, -1/2, -1/2)$. Then by rewriting the Majorana fields as Dirac fields, one readily finds that

$$\pm f_{\alpha\beta}^1 = i \pm f_{\alpha\beta}^2. \quad (5.43)$$

This in turn forces $c_2 = 0$. We can then set $c_1 = 1$ as we are uninterested in an overall normalization.

Lorentz invariance demands

$$f_\pm(\theta_1, \theta_2) = e^{\pm(\theta_1 + \theta_2)/4} f_0(\theta_{12}). \quad (5.44)$$

f_0 is then constrained through 5.2,

$$\begin{aligned} f_0(-\theta) &= -S_2(i\pi - \theta) \frac{2\pi i + 3\theta}{5\pi i - 3\theta} f_0(\theta) \\ &= -S_2(\theta) f_0(\theta), \end{aligned} \quad (5.45)$$

where S_2 is given in 4.22, and through 5.3,

$$\mp i(\pm R_{\beta a}) f_0(\theta - 2\pi i) = f_0(-\theta), \quad (5.46)$$

where $\pm R_{\alpha a} = \pm i$ is the braiding phase. Given that $S_2(\theta)$ can be written as

$$S_2(\theta) = -\exp \left[\int_0^\infty \frac{dx}{x} s\left(\frac{x\theta}{i\pi}\right) G_f(x) \right],$$

$$G_f(x) = \frac{e^{-7x/6} + 2c(x/6)}{c(x/2)}, \quad (5.47)$$

we see that f_0 takes the form

$$f_0(\theta) = \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta)\right) \right]. \quad (5.48)$$

As two kinks of opposite chirality form a fermionic bound state, $\pm f_{\alpha\beta}^a$ should have a pole at $\theta_2 = \theta_1 + iu_{\alpha\beta}^a = \theta_1 + i2\pi/3$. Thus $\pm f_{\alpha\beta}^a$ becomes,

$$\pm f_{\alpha\beta}^a(\theta_1, \theta_2) = A_F (C\gamma^a)_{\alpha\beta} \frac{e^{\pm(\theta_1+\theta_2)/4}}{c(\theta_{12}) + 1/2} \times$$

$$\exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta_{12})\right) \right], \quad (5.49)$$

where A_F is some normalization with mass dimension $[m]^{1/2}$.

To determine the phase of A_F we again employ a hermiticity condition:

$$\pm f_{\alpha\beta}^a(\theta_1, \theta_2)^* = \pm f_{\bar{\beta}\bar{\alpha}}^a(\theta_2 - i\pi, \theta_1 - i\pi). \quad (5.50)$$

This implies the phase of A_F is $e^{\pm i\pi/4}$. Scaling this phase out leaves us with the final form for $\pm f_{\alpha\beta}^a$:

$$\pm f_{\alpha\beta}^a(\theta_1, \theta_2) = A_F e^{\pm i\pi/4} (C\gamma^a)_{\alpha\beta} \frac{e^{\pm(\theta_1+\theta_2)/4}}{c(\theta_{12}) + 1/2} \times$$

$$\exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta_{12})\right) \right]. \quad (5.51)$$

G. Two Particle Form Factors for Kink Operators

At the two particle level, a right/left kink field, ψ_\pm^α , will couple to a kink $A_\beta(\theta)$ of chirality opposite to α together with a Gross-Neveu fermion. So we consider form factors given by

$$\pm f_{\alpha\beta}^\alpha(\theta_1, \theta_2) = \langle \psi_\pm^\alpha(0) A_\beta(\theta_2) A_a(\theta_1) \rangle. \quad (5.52)$$

For $\pm f_{\alpha\beta}^\alpha$ to be invariant under $SO(8)$, it must take the form

$$\pm f_{\alpha\beta}^\alpha(\theta_1, \theta_2) = (C\gamma^a)_{\alpha\beta} f_\pm(\theta_1, \theta_2) + (\gamma^a C)_{\alpha\beta} \tilde{f}_\pm(\theta_1, \theta_2). \quad (5.53)$$

We can set $\tilde{f}_\pm(\theta_1, \theta_2)$ to 0. The constraints upon the form factors 5.2 and 5.3 lead to different functional forms

for $f_\pm(\theta_1, \theta_2)$ and $\tilde{f}_\pm(\theta_1, \theta_2)$. By triality we expect these forms to match that of $\pm f_{\alpha\beta}^a$ and only $f_\pm(\theta_1, \theta_2)$ does.

We must also consider $\pm f_{\beta a}^\alpha$, the form factor obtained from $\pm f_{\alpha\beta}^\alpha$ by reversing the order of particles. By triality both $\pm f_{\alpha\beta}^\alpha$ and $\pm f_{\beta a}^\alpha$ are expressible in terms of form factors of type $\pm f_{\alpha\beta}^a$. Because $\pm f_{\alpha\beta}^a$ is symmetric in α, β , we must then have

$$\pm f_{\alpha\beta}^\alpha = \pm f_{\beta a}^\alpha, \quad (5.54)$$

and so $\pm f_{\beta a}^\alpha(\theta_1, \theta_2) = (C\gamma^a)_{\alpha\beta} f_\pm(\theta_1, \theta_2)$.

Lorentz invariance demands that

$$f_\pm(\theta_1, \theta_2) = e^{\pm(\theta_1+\theta_2)/4} f_0(\theta_{12}). \quad (5.55)$$

This ansatz is again constrained by 5.2 and 5.3 in conjunction with the kink-fermion S-matrix 4.20:

$$f_0(\theta - 2\pi i) = -S_2(\theta) f_0(\theta);$$

$$f_0(\theta - 2\pi i) = f_0(-\theta). \quad (5.56)$$

That these equations match 5.44 and 5.45 is a reflection of triality. Given the bound structure of $\pm f_{\alpha\beta}^a$ is the same as $\pm f_{\alpha\beta}^a$, we can write down

$$\pm f_{\alpha\beta}^a(\theta_1, \theta_2) = \pm f_{\beta a}^\alpha(\theta_1, \theta_2)$$

$$= -A_F e^{\pm i\pi/4} (C\gamma^a)_{\alpha\beta} \frac{e^{\pm(\theta_1+\theta_2)/4}}{c(\theta_{12}) + 1/2}$$

$$\times \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta_{12})\right) \right], \quad (5.57)$$

where the normalization relative to $\pm f_{\alpha\beta}^a$ has been fixed using triality.

H. One Particle Form Factor for Kink Fields

The kink field, ψ_\pm^α , will couple to the anti-kink $C_{\alpha\beta} A_\beta$. Hence we consider the form factor,

$$\pm f_\beta^\alpha(\theta) = \langle \psi_\pm^\alpha(0) A_\beta(\theta) \rangle, \quad (5.58)$$

which must take the form

$$\pm f_\beta^\alpha(\theta) = c_\pm e^{\pm\theta/2} C_{\alpha\beta}. \quad (5.59)$$

To determine the constant, we use the two particle form factor $\pm f_{\alpha\beta}^\alpha$. Let $|\alpha\rangle = (1/2, 1/2, 1/2, 1/2)$ and $|\beta\rangle = (-1/2, -1/2, -1/2, -1/2)$. A_β can be written as

$$ig_{a\gamma}^\beta A_\beta(\theta) = \text{res}_{\delta=0} A_a(\theta + \delta + i\bar{u}_{\gamma\beta}^a) A_\gamma(\theta - i\bar{u}_{a\beta}^{\bar{\gamma}}), \quad (5.60)$$

where $a = 1$, $|\gamma\rangle = (1/2, -1/2, -1/2, -1/2)$. Again the u 's mark out poles in S-matrix indicative of bound states. Here they are given by

$$\bar{u}_{\gamma\bar{\beta}}^a = \bar{u}_{a\bar{\beta}}^{\bar{\gamma}} = \pi - u_{a\bar{\beta}}^{\bar{\gamma}} = \pi/3. \quad (5.61)$$

$g_{a\gamma}^\beta$ can be determined up to a phase from the fermion-kink S-matrix as before to be

$$g_{a\gamma}^\beta = \frac{1}{2} \left(\sqrt{3\pi} \frac{\Gamma(5/3)}{\Gamma(7/6)} \right)^{1/2}. \quad (5.62)$$

Given that,

$$ig_{a\gamma\pm}^\beta f_\beta^\alpha(\theta) = \text{res}_{\delta=0} \pm f_{\gamma a}^\alpha(\theta - i\pi/3, \theta + \delta + i\pi/3), \quad (5.63)$$

together with the hermiticity constraint, $\pm f_\alpha^{\bar{\alpha}}(\theta)^* = (\mp i) \pm f_\alpha^{\bar{\alpha}}(\theta)$, we find (up to a sign)

$$c_\pm = \frac{4}{\sqrt{3}} A_F e^{\pm i\pi/4} \left(\sqrt{3\pi} \frac{\Gamma(5/3)}{\Gamma(7/6)} \right)^{-1/2} \times \exp \left[- \int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} s^2(x/6) \right]. \quad (5.64)$$

I. One Particle Form Factor for the Gross-Neveu Fermions

At the one particle level, the fermion field, ψ_\pm^a will couple only to $A_a(\theta)$. Hence we have

$$\pm f_b^a = \langle \psi_\pm^a(0) A_b(\theta) \rangle = d_\pm e^{\pm\theta/2} \delta_{ab}. \quad (5.65)$$

To determine the constant d_\pm , we use triality. Fixing $a = 2$, it is easy to show under the triality transformation 2.25,

$$\pm f_a^a = \frac{1}{2} (\pm f_\alpha^\alpha + \pm f_\alpha^{\bar{\alpha}}), \quad (5.66)$$

where $\alpha = (1/2, 1/2, 1/2, 1/2)$ and so $d_\pm = c_\pm$.

J. Summary

Here we summarize the results of this section for quick reference:

Two Particle Form Factors:

For the $SO(8)$ currents, G_μ^{ab} , we have

$${}_\mu f_{cd}^{ab}(\theta_1, \theta_2) \equiv \langle G_\mu^{ab}(0) A_b(\theta_2) A_a(\theta_1) \rangle$$

$$= iA_G(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})f_\mu(\theta_1, \theta_2),$$

$$\begin{aligned} {}_\mu f_{\alpha\beta}^{ab}(\theta_1, \theta_2) &\equiv \langle G_\mu^{ab}(0) A_\beta(\theta_2) A_\alpha(\theta_1) \rangle \\ &= i \frac{A_G}{2} (C\sigma^{ab})_{\alpha\beta} f_\mu(\theta_1, \theta_2), \end{aligned} \quad (5.67)$$

where

$$\begin{aligned} f_\mu(\theta_1, \theta_2) &= (e^{(\theta_1+\theta_2)/2} - (-1)^\mu e^{-(\theta_1+\theta_2)/2}) \\ &\times \frac{s(\theta_{12}/2)}{c(\theta_{12}) - 1/2} \\ &\times \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta_{12})\right) \right], \\ G_c(x) &= 2 \frac{c(x/6) - s(x/6)e^{-2x/3}}{c(x/2)}. \end{aligned} \quad (5.68)$$

For the Gross-Neveu fermions, ψ_\pm^a , and the kinks, ψ_\pm^α , we have

$$\begin{aligned} \pm f_{\alpha\beta}^a(\theta_1, \theta_2) &= \langle \psi_\pm^a(0) A_\beta(\theta_2) A_\alpha(\theta_1) \rangle \\ &= A_F e^{\pm i\pi/4} (C\gamma^a)_{\alpha\beta} f_\pm(\theta_1, \theta_2), \\ \pm f_{a\beta}^\alpha(\theta_1, \theta_2) &= \pm f_{\beta a}^\alpha(\theta_1, \theta_2) \equiv \langle \psi_\pm^\alpha(0) A_\beta(\theta_2) A_a(\theta_1) \rangle \\ &= -A_F e^{\pm i\pi/4} (C\gamma^a)_{\alpha\beta} f_\pm(\theta_1, \theta_2), \end{aligned} \quad (5.69)$$

where

$$\begin{aligned} f_\pm(\theta_1, \theta_2) &= \frac{e^{\pm(\theta_1+\theta_2)/4}}{c(\theta_{12}) + 1/2} \\ &\times \exp \left[\int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} \sin^2\left(\frac{x}{2\pi}(i\pi + \theta_{12})\right) \right], \\ G_f(x) &= \frac{2c(x/6) + e^{-7x/6}}{c(x/2)}. \end{aligned} \quad (5.70)$$

One Particle Form Factors:

For the currents, G_μ^{ab} , we have

$$\begin{aligned} {}_\mu f_{\{cd\}}^{ab}(\theta) &\equiv \langle G^{ab}(0) A_{\{cd\}}(\theta) \rangle \\ &= iA_G(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc})(e^\theta - (-1)^\mu e^{-\theta}) \\ &\times \frac{1}{\sqrt{3}} \left(2\sqrt{3\pi} \frac{\Gamma(2/3)}{\Gamma(1/6)} \right)^{-1/2} \end{aligned}$$

$$\times \exp \left[- \int_0^\infty \frac{dx}{x} \frac{G_c(x)}{s(x)} s^2(x/3) \right], \quad (5.71)$$

and for the fermions, ψ_\pm^a , and kinks, ψ_\pm^α ,

$$\begin{aligned} \pm f_b^a &\equiv \langle \psi_\pm^a(0) A_b(\theta) \rangle = c_\pm e^{\pm\theta/2} \delta_{ab}, \\ \pm f_\beta^\alpha &\equiv \langle \psi_\pm^\alpha(0) A_\alpha(\theta) \rangle = c_\pm e^{\pm\theta/2} C_{\alpha\beta}, \end{aligned} \quad (5.72)$$

where the constant c_\pm is

$$\begin{aligned} c_\pm &= \frac{4}{\sqrt{3}} e^{\pm i\pi/4} A_F \left(\sqrt{3\pi} \frac{\Gamma(5/3)}{\Gamma(7/6)} \right)^{-1/2} \\ &\times \exp \left[- \int_0^\infty \frac{dx}{x} \frac{G_f(x)}{s(x)} s^2(x/6) \right]. \end{aligned} \quad (5.73)$$

VI. ACKNOWLEDGMENTS

We would like to thank M.P.A. Fisher, L. Balents, H. Lin, N. MacKay, A. LeClair, D. Duffy, D. Scalapino, G. Sierra, and I. Affleck for useful discussions. This work has been supported by NSERC of Canada and the NSF through the Waterman Award under grant number DMR-9528578 (R.K.) and by the A.P. Sloan Foundation (A.W.W.L.).

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- [1] H. L. Lin, L. Balents, M. Fisher, cond-mat/9801285.
 - [2] F.P. Milliken, C.P. Umbach, and R.A. Webb, Solid State Commun. 97 (1996) 309.
 - [3] P. Fendley, A.W.W. Ludwig, and H. Saleur, Phys. Rev. Lett. 74 (1995) 3005; ibid. 75 (1995) 2196; Phys. Rev. B52 (1995) 8934.
 - [4] E. Dagotto and T. Rice, Science 271 (1996) 618.
 - [5] T.Ebbesen, Phys. Today 49 (1996) 26.
 - [6] R. M. Konik, F. Lesage, A.W.W. Ludwig, and H. Saleur, cond-mat/9806334.
 - [7] L. Balents and M.P.A. Fisher, cond-mat/9611126.
 - [8] Y. Krotov, D. Lee, and S. Louie, cond-mat/9611073.
 - [9] M. S. Dresselhaus, Nature 391 (1998) 19; M. Bockrath, D. Cobden, P. McEuen, N. Chopra, A. Zettl, A. Thess, and R. Smalley, Science 275 (1997) 1922; A. Rao, E. Richter, S. Bandow, B. Chase, P. Eklund, K. Williams, S. Fang, K. Subbaswamy, M. Menon, A. Thess, R. Smalley, G. Dresselhaus, and M. Dresselhaus, Science 275 (1997) 187; J. Wildöer, L. Venema, A. Rinzler, R. Smalley, and C. Dekker, Nature 391 (1998) 59.
 - [10] S. Tans, M. Devoret, H. Dai, A. Thess, R. Smalley, L. Geerligs, and C. Dekker, Nature 386 (1997) 474.
 - [11] S.C. Zhang, Science 275 (1997) 1089.
 - [12] D. Scalapino, S. Zhang, and W. Hanke, cond-mat/9711117.
 - [13] C. Kane, L. Balents, and M. P. A. Fisher, cond-mat/9708054.
 - [14] D. H. Cobden, J. Nygard, M. Bockrath, and P. McEuen, cond-mat/9904179; M. Bockrath, D. H. Cobden, J. Lu, A. G. Rinzler, R. Smalley, L. Balents, and P. McEuen, cond-mat/9812233.
 - [15] P. Azaria, P. Lecheminant, and A. M. Tsvelik, cond-mat/9806099.
 - [16] G. E. Japaridze, A. A. Nersesyan, and P. Wiegmann, Nucl. Phys. B 230 (1984) 511.
 - [17] R. M. Konik, H. Saleur, and A. W. W. Ludwig, cond-mat/0009166.
 - [18] D. Bernard and A. LeClair, Commun. Math. Phys. 142 (1991) 99.
 - [19] V. Emery, S. Kivelson, and O. Zachar, cond-mat/9810155.
 - [20] Hsiu-Hau Lin, Leon Balents, and Matthew Fisher, unpublished.
 - [21] G. Delfino, G. Mussardo, and P. Simonetti, Nucl. Phys. B 473 (1996) 469.
 - [22] F. A. Smirnov, *Form Factors in Completely Integrable Models of Quantum Field Theory*, World Scientific, New Jersey (1992).
 - [23] J. Cardy and G. Mussardo, Nucl. Phys. B 410 (1993) 451.
 - [24] F. Lesage, H. Saleur, and S. Skorik, Nucl. Phys. B 474 (1996) 602.
 - [25] A. LeClair, F. Lesage, S. Lukyanov, and H. Saleur, hep-th/9701022.
 - [26] G. Delfino and G. Mussardo, Nucl. Phys. B 455 (1995) 724.
 - [27] G. Delfino and J. Cardy, hep-th/9712111.
 - [28] J. Cardy and G. Mussardo, Nucl. Phys. B 410 (1993) 451.
 - [29] Leon Balents, private communication.
 - [30] M. Karowski and P. Weisz, Nucl. Phys. B 139 (1978) 455.
 - [31] V. Yurov and A. B. Zamolodchikov, Int. J. Mod. Phys. A 6 (1991) 3419.
 - [32] F. A. Smirnov, Commun. Math. Phys. 132 (1990) 415.
 - [33] D. Bernard and A. LeClair, Nucl. Phys. B 399 (1993) 709.
 - [34] C. Hayward, D. Poilblanc, and D. Scalapino, Phys. Rev. B 53 (1996) 53.
 - [35] Z. Hiroi and M. Takano, Nature 377 (1995) 41.
 - [36] R. Slansky, Physics Reports 79, No. 1 (1981) 1.
 - [37] C. Chamon and E. Fradkin, cond-mat 9612185.
 - [38] I. Affleck and A.W.W. Ludwig, Nucl. Phys. B 330 (1991) 641.
 - [39] I. Affleck, A.W.W. Ludwig, and B. Jones, Phys. Rev. B 52 (1995) 9528.
 - [40] M. Karowski and H. Thun, Nucl. Phys. B 190 (1981) 61.
 - [41] A. Zamolodchikov and A. B. Zamolodchikov, Phys. Lett. B 72 (1978) 481.
 - [42] R. Shankar and E. Witten, Nucl. Phys. B 141 (1978) 349.